Mathematical Reviews

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Mathematical Reviews

Vol. 21, No. 6 June, 1960 Reviews 3313–4088

LOGIC AND FOUNDATIONS

See also 3361, 3946.

3313:

★**Kemeny, John G. A philosopher looks at science.** D. Van Nostrand Co., Inc., Princeton-Toronto-New York-London, 1959. xii+273 pp. $6.50 (text edition $4.95).

In dem vorliegenden Buch wird von berufener Seite eine ausgezeichnete Einführung in die moderne Philosophie der Naturwissenschaften gegeben. Soweit es im Rahmen einer Einführung geschehen kann, werden alle wichtigen und heute diskutierten Probleme behandelt. Die auf den Text bezogenen Literaturhinweise am Ende eines jeden Kapitels verdienen besondere Erwähnung, weil sie so hervorragend ausgewählt sind.

Im ersten Teil des Buches werden die grundlegenden Voraussetzungen der Naturwissenschaften behandelt. Es wird die Rolle der Mathematik als universelle und exakte Sprache der Naturwissenschaften gewürdigt; der Begriff des Naturgesetzes einer eingehenden und interessanten Analyse unterzogen und die Verwendung des Wahrscheinlichkeitsbegriffes in der Naturwissenschaft erörtert.

Im zweiten Teil des Buches untersucht der Verfasser einige Methoden und Begriffe der Naturwissenschaften genauer. Er weist auf den fundamentalen Verfahrenszyklus der Naturwissenschaften hin, der darin besteht, dass man ausgehend von beobachteten Tatsachen via Induktion auf Theorien schliesst, aus diesen per Deduktion Voraussagen herleitet, die dann wieder durch Beobachtungen verifiziert werden können. Die drei grundlegenden Verfahren, nämlich Induktion, Deduktion und Verifikation werden eingehender behandelt. Besonders lesenswert ist das Kapitel 9, in dem der Verfasser allgemeine Kriterien für den Begriff der wissenschaftlichen Erklärung angibt, die an Beispielen diskutiert werden.

Im dritten Teil des Buches werden noch einige allgemeine Fragen behandelt wie Determinismus, das Problem des Lebens, der freie Wille und das Problem der Werte. Alle diese Probleme werden in der klaren und bestechend einfachen Sprache behandelt, die für das ganze Buch charakteristisch ist. *H. Kiesow* (Münster)

3314:

★**Wilson, N. L. The concept of language.** University of Toronto Press, Toronto, 1959. viii+153 pp. $4.95.

The discussion in the book seems to have been inspired largely by writings of Rudolf Carnap. It covers, in the words of the author, only languages of the standard symbolic types, not of the type of used languages. It deals with several difficulties connected with the role played by rules in characterizing a language, with a general definition of designation, with the distinction between semantics and pragmatics. The type of discussion, in spite of the use of terms such as "theorem", does not appear to be what a mathematician would expect.

G. Y. Rainich (Notre Dame, Ind.)

3315:

★**Löb, M. H. Constructive truth.** Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 159–168. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii+297 pp. $8.00.

This paper sketches a formal "constructivist" approach to certain epistemological problems, based upon Kemeny's theory of semantics and the theory of partial recursive functions. Certain concepts called by the author "brain", "subject", "mental occurrence" and "perception" are defined. {There seems to be an error in the definition of "brain".} As an application, the sentence "ξ has a pain in his leg" is given a "formal explication". As to be expected, a good deal of clarification and "detailed extrapolation" by the author would be helpful. The same holds true of the proposed solipsistic notion of "local" formal language and its alleged connection with the semantic paradoxes.

E. Mendelson (New York, N.Y.)

3316:

★**Hilbert, D.; und Ackermann, W. Grundzüge der theoretischen Logik.** 4te Aufl. Die Grundlehren der mathematischen Wissenschaften, Bd. 27. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959. viii+188 pp. DM 33.00.

In the latest edition of this famous text, several substantial changes and improvements are made. In the first place, the method of truth-tables is elaborated more fully. The axiom-systems given throughout are based on those of Gentzen. A new section on intuitionistic sentential logic is added, as well as one on Ackermann's "strenge Implikation". In the chapter on the lower predicate calculus a new section on singular descriptions is added in addition to some material on many-sorted theories. Some of the Hilbert symbols have been replaced, so that '$\neg$' is now the sign for negation, '$\wedge$' for conjunction, and '$\forall x$' and '$\exists x$' for the universal and existential quantifier respectively. Some useful exercises have been appended to the first three chapters. *R. M. Martin* (Austin, Tex.)

3317:

Craig, William. Linear reasoning. A new form of the Herbrand-Gentzen theorem. J. Symb. Logic **22** (1957), 250–268.

Throughout, let A be a conjunction and A' an alternation of first-order formulas in prenex form. The Herbrand-Gentzen theorem can be stated as follows: If $A \supset A'$ is a

theorem of first order predicate calculus and if A is a conjunction and A' is an alternation of prenex normal forms, then $A \supset A'$ can be proved starting from a tautologous matrix and using a certain specific set of rules called H-rules. Each of these applies only to implications and allows us to omit adjacent duplicated terms in the premiss and to introduce quantifiers in appropriate places in the conclusion. There is another set of rules called L-rules which is complete in the sense that if $A \supset A'$ is valid, then A' can be obtained from A by an L-deduction, i.e. by applications of L-rules only. Each L-rule is a one-premiss rule, so that an L-deduction from A to A' is linear, rather than in tree-form. Some of the L-rules are equivalence-rules in which the premiss is equivalent to the conclusion, some are implication-rules where the conclusion is generally weaker than the premiss. Theorem 2 of this paper shows that if $A \supset A'$ is a theorem of first order predicate calculus, then there is an L-deduction from A to A' in which the use of implication-rules is restricted to the middle portion and the use of equivalence rules to the two end-portions and there is in addition a certain symmetry in the deduction. For each such deduction one can construct a corresponding H-deduction (applications of H-rules starting from a tautologous matrix). Thus one obtains a new form of the Herbrand-Gentzen theorem. For each H-deduction the corresponding L-deduction is determined except for inessential differences concerning the order of vacuous quantifiers and variables which do not appear in A or A'. The last important result of this paper states that if $\vdash A \supset A'$ and if A and A' have at least one predicate symbol in common, then there is an intermediate formula B such that $\vdash A \supset B$, $\vdash B \supset A'$, and all predicate symbols occurring in B also occur both in A and in A'. Also if $\vdash A \supset A'$ and if A and A' have no predicate symbol in common, then either $\vdash \neg A$ or $\vdash A'$. The major importance of all these results lies in their applications.

L. N. Gál (New Haven, Conn.)

3318:

Craig, William. Three uses of the Herbrand-Gentzen theorem in relating model theory and proof theory. J. Symb. Logic **22** (1957), 269–285.

The applications discussed here are based on the Herbrand-Gentzen theorem in the form used in the preceding paper, in particular on a strengthened form of theorem 5.

The first is to the relation between the model-theoretic notion of definability of formulas and terms and the corresponding proof-theoretic notion. This extends previous results of Beth [Indag. Math. **15** (1953), 330–339; MR **15**, 385]. Since definability in the proof-theoretic sense generally implies model-theoretic definability, the main results here concern the converse and show that this also holds, and that in fact there is a further extension to second order systems whose extralogical axioms are of the form $(\exists T^1)\cdots(\exists T^m)A$ $(m \geq 0)$, where A is a first order sentence, provided that the expression to be defined and the expressions in terms of which it is to be defined are of first order.

The second application is to the hierarchy of second order formulas and shows that this hierarchy is not analogous to the Kleene hierarchy of recursive sets and their projections.

The third application deals with the problem of axiomatizability of systems with extralogical constants. Given a first-order system and certain subsets of the extralogical constants, the question may arise whether there is an axiomatization in which each extralogical axiom involves only a certain subset of these constants. The last theorem gives a necessary and sufficient condition that this should be possible.

L. N. Gál (New Haven, Conn.)

3319:

Leblanc, Hugues. On logically false evidence statements. J. Symb. Logic **22** (1957), 345–349.

In Carnap's confirmation theory '$c(h, e)$' (read: the degree of confirmation of hypothesis h on evidence e) is regarded as meaningless where 'e' is a logical falsehood. The author gives three ways in which this "restriction", if such it be, may be lifted. The question as to how 'c' is then to be interpreted is left open, so that the advantages of these methods are not clear.

R. M. Martin (Austin, Tex.)

3320:

Prior, A. N. Peirce's axioms for propositional calculus. J. Symb. Logic **23** (1958), 135–136.

In 1885 Peirce set down the following axioms as a basis for the propositional calculus: 1. *Cpp*; 2. *CCpCqrCqCpr*; 3. *CCpqCCqrCpr*; 4. *Cop*; 5. *CCCpqpp*. The present note corrects an inadequacy in the proof that Peirce's basis, with the rules of substitution and detachment, is complete. It is also pointed out that Peirce's proof of the theorem *CNpp* makes use of his axiom 4 as well as the definition of negation as the implication of what is false ($Np = Cpo$). Finally, matrices to show the independence of Peirce's axioms 2, 3, 4, and 5 are given, as well as a proof, based on Łukasiewicz [Ann. Soc. Polon. Math. **22** (1950), supplement, 87–92; reviewed in J. Symb. Logic **20** (1955), 173–174], of the dependence of axiom 1 on the other axioms.

W. J. Feeney (Weston, Mass.)

3321:

Gál, I. L.; Rosser, J. B.; and Scott, D. Generalization of a lemma of G. F. Rose. J. Symb. Logic **23** (1958), 137–138.

A simple proof of the theorem that in the intuitionistic propositional calculus each formula $B \supset C$ is interdeducible with some formula $A \supset C$, where A is a conjunction of terms of the forms a, $\neg a$, $a \supset b$, $a \supset b \vee c$, $a \,\&\, b \supset c$, $(a \supset b) \supset c$. This extends lemma 3.2 of G. F. Rose, Trans. Amer. Math. Soc. **75** (1953), 1–19 [MR **15**, 1], p. 5.

A. Heyting (Amsterdam)

3322:

Lorenzen, P. Logical reflection and formalism. J. Symb. Logic **23** (1958), 241–249.

This paper is concerned with how analysis may be built up along intuitive, constructive, or "operative" lines. "Critical" analysis renounces the use of such concepts as the set of all possible integers. The author argues on behalf of the critical attitude, and shows in detail how the Cantor-Bendixon theorem can be suitably handled on the basis of it.

R. M. Martin (Austin, Tex.)

3323:

Ackermann, Wilhelm. Über die Beziehung zwischen strikter und strenger Implikation. Dialectica **12** (1958), 213–222. (French and English summaries)

In a previous paper [J. Symb. Logic **21** (1956), 113–128;

MR 18, 270] the author discussed a concept of implication which he called "streng" and which will be translated here rather imprecisely as "restricted". His work was motivated by the ideas which brought C. I. Lewis to his notion of strict implication but he managed to avoid some of the anomalies which arise in Lewis' work.

In the present paper, the author correlates his system with that produced by A. Schmidt to formalise Lewis' S2 [Proceedings of the International Congress of Mathematicians, Amsterdam 1954, vol. 2, pp. 407–408, Noordhoff, Groningen, 1957]. He shows that this implication is definable in terms of restricted implication, i.e., there exists a propositional function within the system of restricted implication which satisfies Schmidt's axioms. The question whether it must also satisfy some further sentences which are not provable in Schmidt's system remains open. *A. Robinson* (Jerusalem)

3324:

Schütte, Kurt. Aussagenlogische Grundeigenschaften formaler Systeme. Dialectica **12** (1958), 422–442. (French and English summaries)

If a formal system without types can express all of classical mathematics then one can quickly derive various paradoxes, unless the system is such that not all the laws of propositional calculus hold. Here we have an examination of such systems, their completeness and consistency (appropriately redefined) and relevant decision procedures. *L. N. Gál* (New Haven, Conn.)

3325:

Skolem, Th. Reduction of axiom systems with axiom schemes to systems with only simple axioms. Dialectica **12** (1958), 443–450.

An axiom system of second order S can always be translated into one of first order S'. Relations are regarded here as classes of sequences, i.e., as ordered couples, triples, etc., so that two kinds of fundamental individuals must be recognized in S', the individuals of S and all sequences of such. Properties and relations are then taken as constituting a third domain of individuals in S'. Axioms must be given for S' to assure the existence of a property or relation corresponding to each definable predicate of S.
R. M. Martin (Austin, Tex.)

3326:

Hiż, H. A warning about translating axioms. Amer. Math. Monthly **65** (1958), 613–614.

A translation of a complete axiom-system S into another S' based on another set of primitives is itself complete only if all the definitions of S are provable from the axioms of S'. This point is frequently overlooked, as the author points out in discussing a system of sentential calculus of Halmos.
R. M. Martin (Austin, Tex.)

3327:

★de Bouvère, Karel Louis. A method in proofs of undefinability: With applications to functions in the arithmetic of natural numbers. Dissertation. North-Holland Publishing Co., Amsterdam, 1959. xv+64 pp.

Let T be a deductive theory which, besides the predicates or operations $c_1, c_2, \cdots, c_k$, involves the operation r. Then (as shown by the reviewer in his *L'existence en mathématiques* [Gauthier-Villars, Paris, 1956; MR **19**, 625]) the following method can be applied to show that r is not provably definable in T in terms of $c_1, c_2, \cdots, c_k$: we construct a model M_0 which fulfils the set T_0 of all sentences provable in T and not containing r and which cannot be extended, by introducing a suitable interpretation of r, into a model M which fulfils all sentences provable in T. The author now develops the theory of this method by introducing the notions of compatible definability in T (that is, of provable definability in a suitable extension of T) and of equational definability; the value of these notions is demonstrated by proving a number of theorems and by constructing and discussing a number of examples.
E. W. Beth (Amsterdam)

3328:

Kino, Akiko. A consistency-proof of a formal theory of Ackermann's ordinal numbers. J. Math. Soc. Japan **10** (1958), 287–303.

The author first formalizes Ackermann's system of ordinal numbers [Math. Z. **53** (1951), 403–413; MR **12**, 579] in Gentzen's LK, obtaining a system Γ_A. The proof is carried out by building a model of Γ_A in a subsystem Γ_N of the system G¹LC of Takeuti [Jap. J. Math. **23** (1953), 39–96; MR **17**, 701], and therefore shows the consistency relative to this system. (G¹LC is a Gentzen-type formulation of a certain second order functional calculus.)
L. N. Gál (New Haven, Conn.)

3329:

Rasiowa, H. Sur la méthode algébrique dans la méthodologie des systèmes déductifs élémentaires. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1 (49)** (1957), 223–231.

This paper contains an excellent summary of the methods and results obtained by the application of algebra to logic. In 1950 Rasiowa and Sikorski [Fund. Math. **37** (1950), 193–200; MR **12**, 661] published a new proof of the Gödel completeness theorem for Boolean algebras. There have since been a great many generalizations of this method and many new results. This paper is a summary of these. *L. N. Gál* (New Haven, Conn.)

3330:

★Bernays, Paul. Remarques sur le problème de la décision en logique élémentaire. Le raisonnement en mathématiques et en sciences expérimentales, pp. 39–44. Colloques Internationaux du Centre National de la Recherche Scientifique, LXX. Editions du Centre National de la Recherche Scientifique, Paris, 1958. 140 pp. 1400 francs.

The author sketches Turing's proof of the undecidability of the first-order predicate calculus [Proc. London Math. Soc. **42** (1936), 230–265], but in a more perspicuous form than Turing's original demonstration. The proof shows that the subclass of formulas of the form

$$(x)(Ey)(z)(Eu)(Ev)(Ew)B(x, y, z, u, v, w)$$

is undecidable. In addition, a result of Ackermann concerning satisfiability is a corollary.
E. Mendelson (New York, N.Y.)

3331:

Yablonskiĭ, S. V. Functional constructions in a k-valued logic. Trudy Mat. Inst. Steklov. **51** (1958), 5–142. (Russian)

This paper, actually a monograph, reviews known results and presents refinements of such results in the theory of many-valued logics and its relation to the construction of relay circuits. It comprises three main sections. The first deals with the relation between the functions of algebraic (two-valued) logic and relay circuits. Properties of functions of two-valued logic are defined and discussed, including completeness and closure under composition of a set of such functions, normal forms of functions, and graphical representations of functions having normal forms. Techniques of constructing minimal and reduced normal forms are presented using graphical representations. The last two parts of this section deal with the theory of relay circuits and with functions of two-valued logic as a means for defining n-ary operations in the domain $\{0, 1\}$. Section 2 deals with functional constructions in k-valued logic. Discussed in this section are the notions of homomorphism in a family of functions, duality, closed classes with a finite basis, and the notion of almost-completeness. Criteria for completeness are provided, and the section closes with consideration of a number of functional classes, including monotone, linear, and self dual functions. Section 3 is devoted to the notion of functional completeness in three-valued logics. The main theorem provides necessary and sufficient conditions for a family of such functions to be complete.

E. J. Cogan (Bronxville, N.Y.)

3332:

Hu, Shih-hua; and Loh, Chung-wan. Normal forms of general recursive functions. Acta Math. Sinica **8** (1958), 507–520. (Chinese. English summary)

It is noted that for a wide range of classes K of elementary recursive functions, every recursive function e has a normal form

$$(1)\qquad U(\iota y[t(e, x_1, \cdots, x_n, y) = 0), \quad U, t \text{ in } K,$$

if and only if it has a normal form

$$(2)\qquad \iota y(Ez)h(e, x_1, \cdots, x_n, y, z) = 0, \quad h \text{ in } K.$$

In particular, this is true of Moh's class D [same Acta **4** (1956), 548–564; MR **20** #3070]. A proper subclass A of D is introduced which contains $\overline{\mathrm{sg}}(x)$ (0 only if $x \neq 0$), $\mathrm{sum}(x, y, z)$ (0 only if $x+y=z$), $\mathrm{res}(x, y)$ (0 only if $x=y \dot{-} [\sqrt{y}]^2$), and is closed with respect to (a) substitution including that from g to $f(\cdots, x_i, \cdots)=g(\cdots, x_i', \cdots)$, (b) operation from $f(x_1, \cdots, x_n, i)$ to $f^*(u, x_1, \cdots, x_n, y)$ which is 0 only if $u=\sum_{i=0}^{y} f(x_1, \cdots, x_n, i)$. Functions in A take as values only 0 and 1. The main theorem is that there is a function h in A which gives a normal form of the type (2) above. It follows that for every recursively enumerable set C, there is a function f in A such that $x \in C \equiv (Ey)f(x, y)=0$. Hence, there exists some function g in A such that $(Ey)g(x, y)=0$ gives an undecidable problem. Such results may be useful, e.g., in attempting to get a negative solution to Hilbert's tenth problem; it seems, however, desirable to stress "natural" classes K.

Hao Wang (Murray Hill, N.J.)

SET THEORY

See also 3339.

3333:

Landsberg, Max. Der Durchschnittsgrad hypercharakteristischer Filter. Math. Ann. **131** (1956), 429–434.

Cardinal number $\mathfrak{m}$ is regular [strongly inaccessible] if $\sum \mathfrak{m}_\alpha < \mathfrak{m}$ [$\prod \mathfrak{m}_\alpha < \mathfrak{m}$] whenever $(\mathfrak{m}_\alpha)$ is a family of cardinals indexed by a set of cardinality $< \mathfrak{m}$ such that each $\mathfrak{m}_\alpha < \mathfrak{m}$; $\mathfrak{m}$ is singular if $\mathfrak{m}$ is not regular. A filter on infinite set A is free if it is finer than the filter of complements of all finite subsets of A. (The author uses the terms "accessible," "Tarski," and "hypercharacteristic" respectively for "singular," "strongly inaccessible," and "free.") If Φ is a free filter on A, its intersection degree $d(\Phi)$ is defined to be the smallest of those cardinal numbers $\mathfrak{m}$ such that the intersection of some subfamily of Φ having $\mathfrak{m}$ members does not belong to Φ; clearly $\aleph_0 \leq d(\Phi) \leq \operatorname{card} A$. The author shows that set A admits a free filter satisfying $d(\Phi) = \operatorname{card} A$ if and only if $\operatorname{card} A$ is regular. In the remainder of the paper, the author proves separately special cases of the following theorem: If Φ is a free ultrafilter on A, then $d(\Phi)$ is strongly inaccessible. This theorem follows from slight modifications, given by the author in the proofs of Lemmas 4 and 6, of two arguments of Ulam [Fund. Math. **16** (1930), 140–150] {the author's use of Ulam's matrix argument is unneeded}.

S. Warner (Princeton, N.J.)

3334:

Erdös, P.; Fodor, G.; and Hajnal, A. On the structure of inner set mappings. Acta Sci. Math. Szeged **20** (1959), 81–90.

Let I_1 and I_2 be two arbitrary families of subsets of a given set S. A function G from I_1 into I_2 is called an inner set mapping if $G(X) \subseteq X$ for each element X in I_1. Given an inner set mapping G, for each element X_0 in I_2 let $X_0^{-1} = \bigcup_{G(X)=X_0} X$ and $X_0^{*-1} = \{X \mid G(X) = X_0\}$. An inner set mapping G is said to be of type q and range p if I_1 and I_2 are the families of all subsets of S of powers q and p respectively. Let the symbol $((m, p, q)) \to t$ [resp. $((m, p, q))^* \to t$] mean that for every inner set mapping of type q and range p, defined on a set S of power m, there exists an element X_0 in I_2 for which the power of X_0^{-1} is t [resp. the power of X_0^{*-1} is t].

The paper concerns itself, for the most part, with finding those cardinal numbers m, p, q, and t for which $((m, p, q)) \to t$ is true and those m, p, q, and t for which $((m, p, q))^* \to t$ is true. Using standard techniques of set theory the following are the main results obtained ($q \geq \aleph_0$). (1) If $m^q = q^p$, then $((m, p, q)) \to q^+$ is false and $((m, p, q))^* \to 2$ is false. (2) If $p = q$, then $((m, p, q)) \to q^+$ is false and $((m, p, q))^* \to 2$ is false. (3) If $q^p < m^*$, then $((m, p, q)) \to m$. (4) If $q^p < (m^q)^*$ and $m^p = m^q$, then $((m, p, q))^* \to m^q$. In (1)–(4), t^+ is the cardinal number immediately following t, and t^* is the smallest cardinal number for which t is the sum of t^* numbers each smaller than t.

S. Ginsburg (Hawthorne, Calif.)

3335:

Marcus, Solomon. Sur la représentation d'une fonction arbitraire par des fonctions jouissant de la propriété de Darboux. C. R. Acad. Sci. Paris **249** (1959), 25–26.

Let Y be a set of cardinal a, X a set of cardinal $\geq a$, and

$\mathscr{F}$ a family of cardinal a of sets of cardinal a contained in X. Let D be the class of all functions ψ from X to Y such that $\psi(E)=Y$ for each $E\in\mathscr{F}$, and let f be an arbitrary function from X to Y. There is a sequence $\{f_n\}\subset D$ such that for each $x\in X, f_n(x)=f(x)$ for all but one value of n. If Y is an additive group and p is an integer >1, then f is the sum of p members of D. If Y is a field then f is the product of two functions ψ from X to Y such that $\psi(E)\supset Y-\{0\}$ for each $E\in\mathscr{F}$. These results express the purely set-theoretic content of some theorems of Sierpiński [Matematiche, Catania 8 (1953), no. 2, 43–48, 73–78; MR 16, 229, 230] concerning functions continuous in the sense of Darboux.

J. C. Oxtoby (Bryn Mawr, Pa.)

3336:

Sikorski, R. Some examples of Borel sets. Colloq. Math. 5 (1958), 170–171.

Indicato con H il cubo fondamentale dello spazio di Hilbert, sono costruiti per induzione i seguenti insiemi di Borel M_α ed A_α (in corrispondenza di tutti i successivi indici ordinali α tali che $0\leqq\alpha<\Omega$).—M_0 è il sottoinsieme di H costituito da un unico punto, comunque prefissato, di H, ed è $A_0=H-M_0$. Supposti poi già definiti tutti gli M_ξ ed A_ξ per $\xi<\alpha$, sono distinti due casi. (1) α isolato, cioè $\alpha=\beta+1$: è posto allora $M_\alpha=A_\beta\times A_\beta\times\cdots\subset H^{\aleph_0}$. (2) α non isolato: è posto allora $M_\alpha=A_1\times A_2\times\cdots\times A_\xi\times\cdots=\prod_{\xi<\alpha}A_\xi\subset H^{\aleph_0}$. Essendo $H^{\aleph_0}$ omeomorfo ad H, M_α può essere considerato come sottoinsieme di H e perciò, in entrambi i casi, è lecito porre $A_\alpha=H-M_\alpha$.—Sono dimostrati i seguenti teoremi. (a) M_α è un insieme di Borel esattamente della classe moltiplicativa α in H, ma non della classe additiva α in H [v. per es. F. Hausdorff, *Mengenlehre*, Dover, New York, 1944; MR 7, 419; ivi §§18, 32–34]; A_α è un insieme di Borel esattamente della classe additiva α in H, ma non della classe moltiplicativa α in H. (b) Se X è uno spazio metrico, e $B\subset X$ è un insieme di Borel della classe moltiplicativa [additiva] α in X, esiste un'applicazione continua φ di X su H tale che $\varphi^{-1}(M_\alpha)=B$ [tale che $\varphi^{-1}(A_\alpha)=B$].—Il lavoro termina con alcune interessanti osservazioni critiche.

T. Viola (Torino)

3337:

Bagemihl, Frederick. Some propositions equivalent to the continuum hypothesis. Bull. Amer. Math. Soc. 65 (1959), 84–88.

The paper contains the proofs that 6 propositions, 1–6, are each equivalent to the continuum hypothesis H. Each of these propositions is a conjunction of 2 statements, one dealing with R_1 and the other with R_2 (=plane). The propositions 1, 3, 5 and 2, 4, 6 deal with category and measure, respectively. The category statements are Q_K, Q_K^*, K; those of measure are Q_M, Q_M^* and M; the propositions 1, 3, 5 read: $B_K\wedge Q_K$, $B_K^*\wedge Q_K^*$, $K\wedge Q_K'$; 2, 4, 6 are obtained from 1, 3, 5 by the substitution $K\leftrightarrow M$ (in words, first category [second category]$\leftrightarrow$measure 0 [positive exterior measure]). Here are some specific statements. K: The union of less than c sets of first category of R_1 is of the first category. M^*: R_1 is not a union of less than c sets of measure 0. B_K^*: Let $S\subset R_1$, $T\subset R_1$, card $S<c$, $T\in(I)$; then there exists an $r\in R_1$ such that $S\cap(T+r)=\emptyset$. Q_K: The plane R_2 contains a set A and a family ϕ of horizontal lines such that: (i) the points common to y-axis and $\bigcup\phi$ form a set of the first category; (ii) R_1 contains a set U of second category such that for every $u\in U$ the u-translate of the family ϕ contains a horizontal line intersecting A in $\leqq\aleph_0$ points; (iii) every member of some non-enumerable set of vertical lines intersects $R_2\setminus A$ in $\leqq\aleph_0$ points. The associated propositions K^*, B_K, Q_M are readily obtained by the preceding substitutions $M\leftrightarrow K$. Other statements are of a similar character.

Đ. Kurepa (Zagreb)

3338:

Rubin, Herman. A new form of the generalized continuum hypothesis. Bull. Amer. Math. Soc. 65 (1959), 282–283.

It is shown that the two statements (H) and (*) are equivalent. (H) is the generalized continuum hypothesis, that is, is the logical product of the aleph hypothesis $2^{\aleph_\alpha}=\aleph_{\alpha+1}$ and the axiom of choice. (*) is the statement that, for all transfinite cardinal numbers p and q, if p covers q then $p=2^r$ for some r. (By p covers q is meant that $q<p$, and for no r is $q<r<p$.)

S. Ginsburg (Hawthorne, Calif.)

COMBINATORIAL ANALYSIS

See also 3864, 3910, 4075.

3339:

Harary, Frank. Note on Carnap's relational asymptotic relative frequencies. J. Symb. Logic 23 (1958), 257–260.

This is a short expository note listing asymptotic formulas for the relative frequencies of certain types of relations on a finite set. The relations considered are: symmetric, symmetric irreflexive, reflexive, and transitive irreflexive antisymmetric.

G. Sabidussi (New Orleans, La.)

3340:

Dufresne, Pierre. Relations et triangles servant à la résolution ou à l'illustration des problèmes de dépouillements. C. R. Acad. Sci. Paris 246 (1958), 880–882.

Étant donnée une collection de O objets dont respectivement $a, b, \cdots, n$ appartiennent à des familles $A, B, \cdots, N$, l'auteur se propose de déterminer le nombre de dépouillements satisfaisant à une certaine condition. Il se trouve qu'on peut employer une méthode analogue à celle du triangle de Pascal.

A. Fuchs (Strasbourg)

3341:

Higgins, P. J. Disjoint transversals of subsets. Canad. J. Math. 11 (1959), 280–285.

Let $A_1, A_2, \cdots, A_n$ be a finite collection of subsets of a set A. (The subsets need not be distinct.) The set $\{a_1, a_2, \cdots, a_r\}$ for $r\leqq n$ is a "partial transversal" of $A_1, A_2, \cdots, A_n$ of "length" r provided (i) $a_1, a_2, \cdots, a_r$ are distinct elements of A and (ii) there exists a set of distinct integers $i_1, i_2, \cdots, i_r$ such that a_j is in A_{i_j} for $j=1, 2, \cdots, r$. The author investigates the conditions under which the sets $A_1, A_2, \cdots, A_n$ possess m mutually disjoint partial transversals of prescribed lengths $r_1, r_2, \cdots, r_m$.

Let $r_1, r_2, \cdots, r_m$ be positive integers such that $n\geqq r_1\geqq r_2\geqq\cdots\geqq r_m>0$. Let $[r_i]$ denote the partition formed by the r_i's of $r_1+r_2+\cdots+r_m$. Let $[r_j^*]$ denote the conjugate partition defined by (1) $r_j^*=\sum_{r_i\geqq j}1$ for

$j=1, 2, \cdots, r_1$. If $r_1 < j \leq n$ then define $r_j^* = 0$. Now write (2) $\alpha_k = \sum_{j=n-k+1}^{n} r_j^*$ for $k=1, 2, \cdots, n$. The main theorem asserts the following. "A necessary and sufficient condition for $A_1, A_2, \cdots, A_n$ to have mutually disjoint partial transversals of lengths $r_1, r_2, \cdots, r_m$ is that, for $k=1, 2, \cdots, n$, every k of the A's contain among them at least α_k distinct elements, where α_k is defined by (1) and (2) above." The necessity of this theorem is obvious. The sufficiency, however, requires an intricate induction on n. For $m=1$ and $r_1=n$ the theorem reduces to the P. Hall theorem on representatives of subsets. For $m=1$ and $r_1=r$ the theorem reduces to a theorem of Ore. The theorem is applied to matrices of zeros and ones with prescribed row and column sums and yields a theorem of Gale and the reviewer. *H. J. Ryser* (Columbus, Ohio)

3342:

Walker, Elbert A. Non-linear recursive sequences. Canad. J. Math. **11** (1959), 370–378.

The sequences of the title are $(a_1, a_2, \cdots)$ with a_i either 0 or 1 and $a_{n+i} = f(a_i, \cdots, a_{n+i-1})$ with f an arbitrary Boolean function of n variables. The sequences are periodic and main interest is in those of maximal length, 2^n; N. G. de Bruijn [Nederl. Akad. Wetensch. Proc. Ser. A **49**, 758–764; MR **8**, 247] has shown that the number of maximal sequences is $2^{\mathfrak{f}(n)}$, with $\mathfrak{f}(n) = 2^{n-1} - n$. The function f induces a mapping $F(S)$ on the n-bit word $S = (a_1, a_2, \cdots, a_n)$: $F(S) = (a_2, \cdots, a_n, f(a_1, \cdots, a_n))$; this mapping is a permutation (of 2^n elements) whose cycle structure determines the periodicity of the sequence; a single cycle is required for maximal length. A reverse of a sequence is the sequence read backwards; a dual is the sequence with ones replaced by zeros and vice versa; a reverse-dual = dual-reverse has the obvious definition. If a sequence is of maximal length so are its reverse, dual and reverse-dual sequences, but these need not be distinct. It is shown that if a sequence and its reverse are equal, the sequence is not maximal for $n > 2$ and is associated with an F whose cycle structure contains at least 2^{N-1} cycles, with N the integral part of $(n+1)/2$. It is also shown that if F corresponds to an odd number of cycles then the sequence is not equal to its dual, and if n is even and greater than 2, its reverse is not equal to its dual. An example for $n=5$ shows that for n odd the reverse and dual of a maximal sequence may be equal. *J. Riordan* (New York, N.Y.)

3343:

Bose, R. C.; and Shrikhande, S. S. On the falsity of Euler's conjecture about the non-existence of two orthogonal Latin squares of order $4t+2$. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 734–737.

In 1782 Euler [Verh. Genootsch. der Vlissingen **9** (1782), 85–232] conjectured that there do not exist two orthogonal Latin squares of order $n \equiv 2 \bmod 4$. In this paper this conjecture is broken.

The crucial point of the paper is a generalization of a theorem of E. T. Parker [Amer. Math. Soc. Notices **5** (1958), 815]. Whereas Parker proves that the existence of a balanced incomplete block design with $\lambda=1$ and k a prime power implies the existence of a set of $k-2$ pairwise orthogonal Latin squares of order v, the authors proceed as follows: An arrangement of v elements in b blocks is said to be a pairwise balanced design of index unity and type $(v; k_1, k_2, \cdots, k_m)$, $k_i \leq v$, $k_i \neq k_j$, if each block contains either $k_1, k_2, \cdots$, or k_m distinct elements and every pair of distinct elements occurs in exactly one block of the design. Theorem: If there exists a pairwise balanced design of index unity and type $(v; k_1, k_2, \cdots, k_m)$ and if, further, for each k_i there exist $q_i - 1$ pairwise orthogonal Latin squares of order k_i, then there exist $q-2$ pairwise orthogonal Latin squares of order v, where $q = \min(q_1, q_2, \cdots, q_m)$. The proof of the theorem is effected by showing that on the basis of the assumptions there exists a q by v^2 matrix A whose elements range over a set of v symbols $t_1, t_2, \cdots, t_v$ with the property that any ordered pair of symbols $\binom{t_i}{t_j}$ occurs as a column exactly once in any two-rowed submatrix of A. Using two rows of A to coordinatize, a set of $q-2$ pairwise orthogonal Latin squares of order v is obtained.

A pairwise balanced design of index unity and type $(22; 4, 7)$ is constructed from the balanced incomplete block design with parameters $v=15$, $b=35$, $r=7$, $k=3$, $\lambda=1$ [R. C. Bose, S. S. Shrikhande, and K. N. Bhattacharya, Ann. Math. Statist. **24** (1953), 167–195; MR **15**, 3]. An application of the theorem yields the existence of two orthogonal Latin squares of order 22. Two such squares are exhibited explicitly. *J. K. Goldhaber* (St. Louis, Mo.)

3344:

Parker, E. T. Orthogonal latin squares. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 859–862.

Utilizing an idea of R. C. Bose and S. S. Shrikhande [see the preceding review] the author establishes the existence of pairs of orthogonal Latin squares for infinitely many orders $n \equiv 2 \bmod 4$ and in particular exhibits explicitly two orthogonal Latin squares of order 10.

The following theorem is proved: If q is a prime power and $q \equiv 3 \bmod 4$ then there exists a pair of orthogonal Latin squares of order $n = (3q-1)/2$. A 4 by n^2 matrix A, whose elements range over a set of n symbols, with the property that any ordered pair of symbols occurs as a column exactly once in any two rowed submatrix of A, is constructed as follows: Consider the ordered quadruples $(X_i, a, r^{2i}+a, r^{2i}(r+1)+a)$, where $i = 1, 2, \cdots, (q-1)/2$, a ranges over the elements of GF $[q]$, r is a fixed primitive element of GF $[q]$, and the X_i are symbols, together with the quadruples obtained from these by cyclic permutation of the four positions. Supplement these $2q \cdot (q-1)$ ordered quadruples by (a, a, a, a), where a ranges over the elements of GF $[q]$, and by a set of $(q-1)^2/4$ ordered quadruples equivalent to a pair of orthogonal Latin squares of order $(q-1)/2$, with the X_i in all four positions. The matrix whose columns consist of the above ordered quadruples is shown to have the aforementioned property. By coordinatization, two orthogonal Latin squares of order n are obtained.

Since there are infinitely many primes $q \equiv 7 \bmod 8$ it follows that there are pairs of orthogonal Latin squares for infinitely many orders $n \equiv 2 \bmod 4$. With $q=7$ the theorem yields two orthogonal Latin squares of order 10.

Using a method of MacNeish [Ann. Math. **23** (1921/22), 221–227] the following corollary is obtained: There exists a pair of orthogonal Latin squares of order any odd multiple of any order constructed in the theorem. *J. K. Goldhaber* (St. Louis, Mo.)

3345:

Isbell, J. R. An inequality for incidence matrices. Proc. Amer. Math. Soc. 10 (1959), 216–218.

Let v, w, λ, k, l be positive integers. We consider a finite set of v elements, and there are w distinct subsets which are designated as "lines".

Theorem 1: If every two lines have at least λ common points, if no line contains more than k points and if each point belongs to exactly l lines, then we have $\lambda(v-1) \leq k^2-k$. There is equality only if we have a (v, k, λ) configuration (then $v=w$, $l=k$, each line has exactly k points, any two lines have exactly λ points in common, and any two points are joined by exactly λ lines; some of these assumptions follow from the others [see S. Chowla and H. J. Ryser, Canad. J. Math. 2 (1950), 93–99; MR 11, 306]).

Theorem 2: If each pair of lines has exactly λ points in common, then $w \leq v$. This generalizes the case $\lambda=1$, which is due to P. Erdös and the reviewer [Nederl. Akad. Wetensch. Proc. 51 (1948), 1277–1279; MR 10, 424].

N. G. de Bruijn (Amsterdam)

3346:

Atiquallah, M. Some new solutions of symmetrical balanced incomplete block design with $\lambda=2$ and $k=9$. Bull. Calcutta Math. Soc. 50 (1958), 23–28.

The present paper studies solutions for a symmetrical balanced incomplete block design with parameters $v=b=(n^2+n+2)/2$, $r=k=n+1$, $\lambda=2$, when $n=8$. The author obtains four more independent solutions different from the two discovered by Hussain [same Bull. 37 (1945), 115] and from the solution given by Bose [Ann. Eugenics 9 (1939), 353], Fisher and Yates [Stat. Tables for Biol., Agric. and Med. Research, 1938, 1942].

H. J. Ryser (Columbus, Ohio)

3347:

Erdös, P. Remarks on a theorem of Ramsay. Bull. Res. Council Israel. Sect. F 7F (1957/58), 21–24.

Let $f(k, l)$ be the least integer so that every graph of $f(k, l)$ vertices contains either a complete graph (every pair of vertices connected by an edge) of k vertices or a set of l independent vertices (no pair of vertices connected by an edge). $f(k, l)$ has been determined for small values of k and l [Greenwood and Gleason, Canad. J. Math. 7 (1955), 1–7; MR 16, 733; Hanani, oral communication] and it is known that

$$f(k, l) \leq \binom{k+l-2}{k-1}$$

[Erdös and Szekeres, Compositio Math. 2 (1935), 463–470] and $2^{k/2} \leq f(k, k)$ [Erdös, Bull. Amer. Math. Soc. 53 (1947), 292–294; MR 8, 479]. In this note the author describes a graph which establishes $\lim_{l\to\infty} f(3, l)=\infty$ and then proves the stronger result $f(3, l) > l^{1+c}$, where $c>0$. This implies $\phi(k) < k^{c_1}$, where $\phi(k)$ is the smallest integer for which there exists a k-chromatic graph of $\phi(k)$ vertices which does not contain a triangle.

W. Moser (Winnipeg, Man.)

3348:

Clarke, L. E. On Otter's formula for enumerating trees. Quart. J. Math. Oxford Ser. (2) 10 (1959), 43–45.

Let t_n and T_n be respectively the number of free and rooted trees having n points, and let $t(x)$ and $T(x)$ be the corresponding generating functions. Otter [Ann. of Math. (2) 49 (1948), 583–599; MR 10, 53] has shown that

$$t(x) = T(x)-(T^2(x)-T(x^2))/2.$$

The present paper gives an elementary proof of this formula using the centroid of a tree. Other proofs of Otter's result are given by Riordan [*An introduction to combinatorial analysis*, Wiley, New York, 1958; MR 20 #3077] and the reviewer [Michigan Math. J. 3 (1955/56), 109–112; MR 17, 1231], while previous solutions for t_n in terms of the known values of T_n appear in Pólya [Acta Math. 68 (1937), 145–254] and Cayley [*Collected mathematical papers*, vol. 11, Cambridge University Press, 1896; paper 772].

F. Harary (Ann Arbor, Mich.)

3349:

Sabidussi, Gert. On the minimum order of graphs with given automorphism group. Monatsh. Math. 63 (1959), 124–127.

Corresponding to any finite group G there exist finite graphs whose automorphism group is isomorphic to G [R. Frucht, Canad. J. Math. 1 (1949) 365–378; MR 11, 377]. Among these graphs there is one, or more, having the least number of vertices; this number is denoted by $\alpha(G)$. The purpose of this paper is to estimate $\alpha(G)$. It is proved that $\alpha(G)=O(m \log n)$, m being the order of G and n the minimal number of generators of G. Since $n=O(\log m)$ it follows that $\alpha(G)=O(m \log \log m)$. The author points out that this result is far from being best possible. In addition, $\alpha(Z_m)$, where Z_m is the cyclic group of order m, is evaluated.

G. A. Dirac (Hamburg)

ORDER, LATTICES

See also 3329, 3492, 3500, 3533, 4018.

3350:

Choe, Tae Ho. An isolated point in a partly ordered set with interval topology. Kyungpook Math. J. 1 (1958), 57–59.

Let P be a partly ordered set with the interval topology. That is, the set of all $L(a)=\{x: s \leq a\}$ and $M(a)=\{x: x \geq a\}$ is a subbasis for the closed sets. An element a of a subset M of P is isolated in M if and only if there exist finite subsets A and B of P such that $A=\{x: x>a$ or x is not comparable with $a\}$, $B=\{y: y<a$ or y is not comparable with $a\}$, and $M \setminus \{a\}=(\bigcup_{x\in A} M(x)) \cup (\bigcup_{y\in B} L(y))$. This generalizes a result of Northam for lattices [Proc. Amer. Math. Soc. 4 (1953), 824–827; MR 15, 244; see p. 827].

P. F. Conrad (New Orleans, La.)

3351:

Kurepa, Gjuro. On two problems concerning ordered sets. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II 13 (1958), 229–234. (Serbo-Croatian summary)

(1) Any nondenumerable (partially) ordered set in which every antichain is finite contains a nondenumerable chain. (2) There exists a nondenumerable ordered set, all of whose nondenumerable subsets contain infinite chains, but which contains no nondenumerable chain. The author proves these and some related statements. He points out that (1) and (2) answer the questions raised by Sierpiński

in *Cardinal and ordinal numbers* [Państwowe Wydawnictwo Naukowe, Warsaw, 1958; MR **20** #2288; pp. 190–191] and that the problems had already been solved by Dushnik and Miller, and Kurepa, respectively.

L. Gillman (Princeton, N.J.)

3352:

Utumi, Yuzo. On a theorem on modular lattices. Proc. Japan Acad. **35** (1959), 16–21.

An irreducible, complete, continuous, complemented modular lattice L is known to have finite dimension if and only if: (N) L contains no infinite sequence $(a_1, a_2, \cdots)$, all $a_i \neq 0$, such that $a_{i-1} \geqq a_i \dot{\cup} b_i$ (direct union) and $a_i \approx b_i$ (a_i, b_i projective) for every $i > 1$ and some $b_i \in L$. It is shown in this paper that for a complete, upper continuous modular lattice L condition (N) is equivalent to each of conditions (M) and (F) stated below. (M) $m(L)$ is finite, where $m(L)$ is the least upper bound of the set of all integers r for which L contains an independent set of r mutually projective elements. (F) There is no countable independent set $(a_1, a_2, \cdots) \subset L$ such that for each i there exists some $b_{i+1} \leqq a_i$ having the property that $b_{i+1} \approx a_{i+1}$. Let R be a semisimple ring with unity. The result above is used to prove that if R is injective as a left R-module, then: (i) R has bounded index; (ii) R/A is a simple ring with minimum condition for every primitive ideal A; (iii) R is P-soluble in the sense of Levitzki; (iv) R is injective as a right R-module.

R. E. Johnson (Northampton, Mass.)

3353:

Szász, Gábor. On the structure of complemented lattices. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **9** (1959), 57–79. (Hungarian)

Hungarian version of the papers in Acta Sci. Math. Szeged **18** (1957), 48–51; **19** (1958), 77–81, 224–228; Publ. Math. Debrecen **5** (1958), 217–221 [MR **19**, 380; **20** #2293, #6373, #6986].

L. Fuchs (Budapest)

3354:

Papert, S. Which distributive lattices are lattices of closed sets? Proc. Cambridge Philos. Soc. **55** (1959), 172–176.

A one-to-one mapping of a complete lattice L into the lattice of all subsets of a set is called a Λ-isomorphism if it preserves arbitrary products and finite sums. The existence of such a mapping is shown to be equivalent to a weakened form of the general distributive law.

B. Jónsson (Minneapolis, Minn.)

3355:

Heider, L. J. Prime dual ideals in Boolean algebras. Canad. J. Math. **11** (1959), 397–408.

The main results of this paper consist of characterizations of Boolean algebras B that are isomorphic to the field of all open and closed subsets of, respectively, P-spaces, P'-spaces, and U-spaces in the sense of L. Gillman and the reviewer [Trans. Amer. Math. Soc. **77** (1954), 340–362; **82** (1956), 362–391; MR **16**, 156; **18**, 9]. The characterizations involve completeness assumptions, and properties of the dual prime ideals of B.

M. Henriksen (Lafayette, Ind.)

3356:

Krasner, Marc. Les algèbres cylindriques. Bull. Soc. Math. France **86** (1958), 315–319.

A summary of the results by M. Krasner [J. Math. Pures Appl. **18** (1939), 417–418, *Algèbre et théorie des nombres*, Colloq. Internat. C.N.R.S. no. 24, pp. 163–168, Centre National de la Recherche Scientifique, Paris, 1950; MR **1**, 198; **12**, 796] and J. Sebastião e Silva [Pont. Acad. Sci. Comment. **9** (1945), 327–357; MR **10**, 348] which use cylindric algebras in an abstract generalization of some theorems in Galois theory.

B. A. Galler (Ann Arbor, Mich.)

3357:

★Nolin, L. Sur l'algèbre des prédicats. Le raisonnement en mathématiques et en sciences expérimentales, pp. 33–37. Colloques Internationaux du Centre National de la Recherche Scientifique, LXX. Editions du Centre National de la Recherche Scientifique, Paris, 1958. 140 pp. 1400 francs.

The author constructs a model for the calculus of predicates more accessible to the professional algebraist than those provided by Tarski and Halmos. In his own words, he attempts "to construct a relatively simple system whose axioms are equalities and which can be easily interpreted as the classical algebra of first order predicates." The algebra is a set of operations on n-ary predicates which includes introduction of fictitious variables, permutation of variables, replacement of a variable by a constant, replacement of all occurrences of one variable by another variable, disjunction of a kind, and complementation. Other operations are compounded from these. Remarks by Mostowski, Tarski, Porte, and Riquet are included.

E. J. Cogan (Bronxville, N.Y.)

3358:

★Bernays, Paul. Über eine natürliche Erweiterung des Relationenkalküls. Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 1–14. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii+297 pp. $8.00.

In the Tarski-Thompson axioms for cylindric algebras the axiom of locally finite dimensionality is not expressed by a first order formula. To avoid this situation the author proposes a system of ten axioms, or axiom schemata, that start with an infinite sequence of underlying sets, with the kth set corresponding to relations in k variables or less. The sufficiency of the axioms is established with the aid of the known representation theorem for cylindric algebras.

B. Jónsson (Minneapolis, Minn.)

GENERAL MATHEMATICAL SYSTEMS

3359:

Chang, C. C.; and Morel, Anne C. On closure under direct product. J. Symb. Logic **23** (1958), 149–154.

It is shown that a class of properties, which A. Horn [same J. **16** (1951), 14–21; MR **12**, 662] showed to be preserved under direct product, does not contain all first order properties thus preserved. First, it is shown that every Horn property is preserved under the formation of (a

special case of) "reduced product": A' is obtained from the direct product A of infinitely many factors A_i by identifying elements that differ in only a finite number of components. Second, it is observed that a certain property, which for Boolean algebras reduces to the existence of at least one atom, is preserved under direct product, while this property is not preserved under reduced product, and hence can not be a Horn property.
R. C. Lyndon (Ann Arbor, Mich.)

3360:

Brunovský, P. Über verallgemeinerte algebraische Systeme. Acta Fac. Nat. Univ. Comenian. **3**, 41–54 (1958). (Slovak. Russian and German summaries)

Unter einer verallgemeinerten (n-aren) Operation f_α in einer Menge A versteht der Verf. eine Funktion, welche jeder Folge von $n=n(\alpha)$ Elementen aus A eine Teilmenge der Menge A zuordnet. Wenn in A eine Menge $\{f_\alpha\}$ von verallgemeinerten Operationen definiert ist, dann heisst A eine Multialgebra. Die Begriffe: homomorphe Abbildung, reguläre Zerlegung, Faktor-Multialgebra werden in einer natürlichen Weise eingeführt. Wenn R_i reguläre Zerlegungen sind, dann ist auch die Zerlegung $\mathbf{V}R_i$ regulär; ein ähnlicher Satz für $\mathbf{\Lambda}R_i$ gilt nicht. Es werden drei Sätze über homomorphe Abbildungen bewiesen, welche mit den bekannten Sätzen der Gruppentheorie analog sind.
J. Jakubík (Košice)

3361:

McLain, D. H. Local theorems in universal algebras. J. London Math. Soc. **34** (1959), 177–184.

This paper proves two metamathematical theorems concerning algebraic systems, which include as special cases such known results as the theorem stating that being a generalized solvable group is a local property. A local system of subalgebras of an algebra A is a family $\mathscr{L}$ of subalgebras having the property that every finite subset of A is contained in some member of $\mathscr{L}$. One of the theorems referred to states that for certain specified types of second order properties P the following holds: Suppose $\mathscr{L}$ is a local system of subalgebras of A and $\mathscr{G}=\cdots\subseteq G_\alpha\subseteq\cdots$ is a complete chain of subalgebras of A. Then $\mathscr{G}$ has a refinement with the property P if and only if, for each $B\in\mathscr{L}$, the chain $\mathscr{G}_B=\cdots\subseteq G_\alpha\cap B\subseteq\cdots$ has such a refinement.
B. Jónsson (Minneapolis, Minn.)

3362:

Nerode, A. Composita, equations, and freely generated algebras. Trans. Amer. Math. Soc. **91** (1959), 139–151.

The concept of a "compositum" is an abstraction of that of an algebra T free on a set V of generators. Precisely, for V a non-empty set, a V-compositum consists of any semigroup S of maps, including the identity, from a set T, containing V, into itself, which has the property that each map of V into T has a unique extension in S. In a "concrete compositum" T is the set of all maps from R^I into R, for some non-empty R and I, V is the set of projections $v_i(r_1, r_2, \cdots)=r_i$, and S is the set of all substitutions $s: T\to T$ defined by the condition

$$(st)(r_1, r_2, \cdots) = t((sv_1)(r_1, r_2, \cdots), (sv_2)(r_1, r_2, \cdots), \cdots).$$

Appropriate definitions, which we do not repeat here, are given for homomorphisms, for subcomposita, and for congruences on composita. The "regular representation" obtained by letting a compositum T operate on itself by composition embeds T in a concrete compositum; it is shown that T is isomorphic to some concrete compositum if and only if there is more than one constant (element of T invariant under S) and the representation of T by operation on the set of constants is faithful. A "completeness theorem" gives, for every proper congruence C, a concrete representation with kernel C.

Abstract algebras in the sense of Birkhoff are associated naturally with corresponding composita, and by these means the equivalence is established between the characterization of free algebras in terms of identities and that in terms of the extendibility of maps. A method of Birkhoff is used to construct an algebra free on a given set of generators, with the same identities as a given algebra; another idea of Birkhoff is extended to discuss a kind of subdirect reduction under "fully invariant" homomorphisms.
R. C. Lyndon (Princeton, N.J.)

3363:

Marczewski, E. A general scheme of the notions of independence in mathematics. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 731–736.

A subset N of an algebra $\mathfrak{A}=\langle A, F\rangle$ is said to be independent provided every mapping of N into A can be extended to a homomorphism of the algebra generated by N into $\mathfrak{A}$. Several known concepts are shown to be special cases of this notion: Linear and algebraic independence of numbers, linear independence of vectors, algebraic independence of polynomials, logical independence of formulas, independence of sets, and independence of Boolean polynomials.
B. Jónsson (Minneapolis, Minn.)

3364:

Świerczkowski, S. On independent elements in finitely generated algebras. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 749–752.

The following three theorems are proved. (1) If a finite n-generated algebra contains n independent elements, then for any set of n elements of A to generate A it is necessary and sufficient that they be independent. (2) If an algebra A is generated by n independent elements and also by $n+1$ independent elements, then it is generated by m independent elements for every $m>n$. (3) If an n-generated algebra A contains $n+1$ independent elements, then it contains an infinite set of independent elements.
B. Jónsson (Minneapolis, Minn.)

3365:

Kuroš, A. G. Direct decompositions in algebraic categories. Trudy Moskov. Mat. Obšč. **8** (1959), 391–412. (Russian)

The author develops a theory of direct decompositions in non-abelian categories far enough to conclude with an interesting partial analysis of the theorem of Remak for finite groups. [For the abelian case cf. M. Atiyah, Bull. Soc. Math. France **84** (1956), 307–317; MR **19**, 172.] The frame is an abstract category in which certain families of mappings $\alpha_i: A\to B$ are "summable" by a commutative associative addition over which composition distributes; for every A and B there is a zero mapping $\omega: A\to B$; every mapping $\alpha: A\to B$ has a kernel $\mu: K\to A$; and the formal

properties of these constructs include a restricted cancellation law but (naturally) not the existence of $-\alpha$. Then the mappings $A \to A$ form a system which the author calls a "semiring".

A monomorphism is defined as a right cancellable mapping (in a more common notation, left cancellable). Then a normal monomorphism is defined as in S. MacLane, Bull. Amer. Math. Soc. **56** (1950), 485–516 [MR **14**, 133]. Such propositions as normality of every kernel, and essential identity of internal and external direct sum decompositions, then go through as usual. Algebraic criteria for existence of common refinements and for isomorphism of two decompositions are also as usual. Finally the theorem of Remak is reduced to the proposition that if α and β are projections upon indecomposable direct summands then $\alpha\beta\alpha$ either is nilpotent or induces an automorphism of the image of α.

J. Isbell (Lafayette, Ind.)

THEORY OF NUMBERS

See also 3470, 3812.

3366:

Thébault, Victor. Questions d'arithmétique. Mathesis **67** (1958), 249–257.

Elementary observations concerning the number of digits, in base B, of the terms of a sequence $\{u_n\}$ satisfying $u_{n+1} = u_n + u_{n-1}$.

Numbers a, b, c and bases B are found such that, in base B, $abba = (cc)^2$, and one of a, b, c is the arithmetic mean of the other two. For example, in base $B = 3739$, $\overline{1429}\ \overline{547}\ \overline{547}\ \overline{1429} = (\overline{2311}\ \overline{2311})^2$.

Numbers a, b, c, d $(c \neq d)$ and bases B are found such that, in base B, $(cd)^2 = aabb$ with b a perfect square. For example, in base $B = 360$, $(\overline{46}\ \overline{198})^2 = 66\ \overline{324}\ \overline{324}$.

L. Moser (Edmonton, Alta.)

3367:

Teĭtel'baum, V. N. Comparison of numbers in the Czech system of numbers. Dokl. Akad. Nauk SSSR **121** (1958), 807–810. (Russian)

A system of numbers is called a Czech system when the numbers are given by their least residues modulo the prime numbers $p_1, \cdots, p_n$. The article gives a method to compare the magnitude of two numbers given in this way when: (1) the numbers are smaller than the product $M = p_1 \cdots p_n$; (2) the number 2 is among the basic numbers.

W. H. Muller (The Hague)

3368:

Svenson, Erik. Operatorentheorie der unendlichen periodischen Bruchentwicklungen. Math. Nachr. **19** (1958), 100–128.

For integral base n, $|n| > 1$, let $(a)_n$ be the n-ary representation of a. Let $\{m\}_n$ be the set of all $(r/m)_n$, $r = 1, \cdots, m$. The effect on $\{m\}_n$ of simple changes of m or n is described.

T. S. Motzkin (Los Angeles, Calif.)

3369:

Comment, P. Sur l'équation fonctionelle

$$F(nm)F((n, m)) = F(n)F(m)f((n, m)).$$

Bull. Res. Council Israel. Sect. F **7F** (1957/58), 14–20.

Given a completely multiplicative function f $(f(1) = 1$ and $f(mn) = f(m)f(n)$ for all positive integers m and $n)$ the author considers the problem of finding all functions F which satisfy the equation

$$(1) \qquad F(mn)F((m, n)) = F(m)F(n)f((m, n)),$$

where (m, n) denotes the g.c.d. of m and n. Anderson and Apostol [Duke Math. J. **20** (1953), 211–216; MR **14**, 951] have shown that an infinity of solutions is given by the formula

$$F(n) = \sum_{d|n} f(d)g(n/d)\mu(n/d),$$

where g is an arbitrary multiplicative function $(g(1) = 1$ and $g(mn) = g(m)g(n)$ whenever $(m, n) = 1)$ and μ is the Möbius function. In particular, when $f(n) = n$ and $g(n) = 1$ the corresponding F is Euler's ϕ-function.

The author denotes by E_f^0 the set of functions F which satisfy (1) and such that $F(1) \neq 0$. Various properties of the set E_f^0 are derived. For example, his first theorem states that $E_f^0 = (QM) \cap (K_f)$, where (QM) is the set of all F such that $F(1) \neq 0$ and $F(mn)F(1) = F(m)F(n)$ for $(m, n) = 1$, and (K_f) is the set of all F which satisfy the equation $F(p^\alpha) = F(p)f(p)^{\alpha-1}$ for all primes p and all $\alpha > 1$.

T. M. Apostol (Pasadena, Calif.)

3370:

Harris, V. C.; and Warren, Leroy J. A generating function for $\sigma_k(n)$. Amer. Math. Monthly **66** (1959), 467–472.

Let $\sigma_k(n)$ denote, as usual, the sum of the kth powers of the divisors of the positive integer n. The authors investigate conditions under which a function $f_k(an+b)$ exists, such that the (formal) representation

$$(*) \qquad \sum_{n=1}^{\infty} \sigma_k(an+b)x^{an+b} = \sum_{n=1}^{\infty} \frac{f_k(an+b)x^{an+b}}{1-x^{an+b}}$$

holds, where a is a given positive, and b a given nonnegative, integer. The main conclusion is that such functions exist if, and only if, a divides b. The functions $f_k(an+b)$, when they exist, are obtained explicitly. A discussion of several examples concludes the paper.

A. Sklar (Chicago, Ill.)

3371:

Jarden, Dov. Divisibility of terms by their subscripts in sequences of sums of powers. Riveon Lematematika **12** (1958), 78–79. (Hebrew)

Let $x_1, \cdots, x_s$ be the roots of the equation

$$f(x) = x^s + a_2x^{s-2} + \cdots + a_s = 0,$$

where $a_2, \cdots, a_s$ are integers. We shall say that an integer n has property A_f if $x_1^n + \cdots + x_s^n \equiv 0 \pmod n$. The primes have property A_f for every f.

An interesting unsolved problem is whether any A_f or any (finite) combination of A_f characterizes the primes. The author observes that not every A_f characterizes the primes. (Indeed it is obvious that every n satisfies infinitely many A_f for any prescribed degree s.) Even the stronger condition $x_1^{n^i} + \cdots + x_s^{n^i} \equiv 0 \pmod n$ for all $i = 1, 2, \cdots$ does not characterize primes for arbitrary f.

E. G. Straus (Los Angeles, Calif.)

3372:

McAndrew, M. H. Note on a problem of Erdös. Proc. Cambridge Philos. Soc. **55** (1959), 210–212.

The author proves the following theorem. Given

non-negative integers $a_1, \cdots, a_n$ and $c_1, \cdots, c_n$ such that $a_1 \neq 0$ and $c_1 = 0$, then there exist infinitely many integers x such that the quotient

$$\left(x \sum_{i=1}^{n} a_i\right)! \Big/ \prod_{i=1}^{n} \{(a_i x + c_i)!\}$$

is an integer. This result was first proved by E. M. Wright [J. London Math. Soc. **33** (1958), 476–478; MR **21** #18] with the added hypothesis that $\sum a_i$ is a prime. Special cases were treated by Mordell [ibid. **34** (1959), 134–138; MR **21** #17] and Erdös [Amer. Math. Monthly **54** (1947), 286]. *T. M. Apostol* (Pasadena, Calif.)

3373:

Estrugo, José Antonio. Simultaneous divisibility criteria for sets of prime numbers. Gac. Mat. Madrid (1) **10** (1958), 204–214. (Spanish)

Let N and P be natural numbers with $P \equiv \pm 1 \pmod{10}$. The author explains a simple algorithm for dividing N by P: $N = QP + R$, $0 \leq R < P$. The condition that N be divisible by a prime power factor of P is, then, that R be divisible by that prime power. If a set of prime powers is given, simultaneous criteria for divisibility by these are thus obtained by taking P equal to their product, or their product multiplied by 3; powers of 2 or 5 are, of course, to be omitted. *H. W. Brinkmann* (Swarthmore, Pa.)

3374:

Sierpiński, W. Sur une question concernant le nombre de diviseurs premiers d'un nombre naturel. Colloq. Math. **6** (1958), 209–210.

Let $\nu(n)$ be the number of distinct prime factors of n. It is shown simply that if $\nu(n) = \nu(n+1) = 1$ and $n \neq 8$ then n is a Mersenne prime or $n+1$ is a Fermat prime. This was previously shown in essentially the same way by C. T. C. Wall [Eureka no. 19 (1957), 10–11; MR **18**, 717]. It is not known whether there are infinitely many solutions of $\nu(n) = \nu(n+1)$, though according to a conjecture of Schinzel there exist infinitely many solutions to $\nu(n) = \nu(n+1) = \cdots = \nu(n+k)$ for every k. *L. Moser* (Edmonton, Alta.)

3375:

Wright, E. M. Prouhet's 1851 solution of the Tarry-Escott problem of 1910. Amer. Math. Monthly **66** (1959), 199–201.

The author gives historical proof of the fact that the problem generally known as the Tarry-Escott problem was anticipated in more general form by Prouhet by the amount indicated in the title. Prouhet's result may be stated as follows. Consider all j^k integers $n \geq 0$ that have k (or fewer) digits when written in the scale of j. Two such integers are considered equivalent if the sums of their digits are congruent modulo j. Then the j equivalence classes are such that the sums of the mth powers of the members of each class are all the same, this for $m = 0(1)k-1$. The author gives a new proof of this using the finite difference operator $E = 1 + \Delta$ and jth roots of unity. [For three other proofs by the reviewer and the author see Lehmer, Scripta Math. **13** (1947), 37–41; MR **9**, 78; Wright, Proc. Edinburgh Math. Soc. (2) **8** (1949), 138–142; MR **12**, 11]. *D. H. Lehmer* (Berkeley, Calif.)

3376:

Grosswald, E.; Calloway, A.; and Calloway, J. The representation of integers by three positive squares. Proc. Amer. Math. Soc. **10** (1959), 451–455.

Theorem: There exists a finite set S of numbers with the following property. If $n > 0$ is not divisible by 4, not $\equiv -1 \pmod 8$ and not in S, then n is the sum of three strictly positive squares. The authors conjecture that S can be taken to consist of 1, 2, 5, 10, 13, 25, 37, 58, 85, 130 only. The proof, which is short, depends on Siegel's estimate for $L(1, \chi)$. The authors also give an asymptotic formula for the number of integers less than x which are the sum of three strictly positive squares, but this formula depends on the set S. There is a review of earlier work on the analogous problem for s squares ($s \geq 4$). *J. W. S. Cassels* (Cambridge, England)

3377:

Cassels, J. W. S. Note on quadratic forms over the rational field. Proc. Cambridge Philos. Soc. **55** (1959), 267–270.

The author considers the following theorem. Let $f(x)$ be a non-singular quadratic form in n variables with coefficients in R. Suppose that f represents 0 in R_p for every prime p and in R_∞. Then f represents 0 in R, the field of rational numbers. Except for $n=4$ the proof of this theorem requires no deep tools but for $n=4$ the truth of the theorem has been shown only by use of fairly deep results such as Dirichlet's theorem on primes in an arithmetic progression. The author gives a simple proof using nothing deeper than Minkowski's convex body theorem and Gauss's theorem on the existence of forms in genera. *B. W. Jones* (Mayaguez, P.R.)

3378:

Mordell, L. J. Integer solutions of simultaneous quadratic equations. Abh. Math. Sem. Univ. Hamburg **23** (1959), 126–143.

The author proves the following theorem. Let

$$f_n(x) = f_n(x_1, x_2, \cdots, x_n), \quad g_n(x) = g_n(x_1, x_2, \cdots, x_n)$$

be two non-singular quadratic forms with rational coefficients in n variables. Suppose that for real λ, μ, no member of the pencil $\lambda f_n(x) + \mu g_n(x)$ is either definite of rank $\leq n$, or singular of rank < 5, but that at least one member has more than four positive and more than four negative squares when expressed as a sum of squares of real linear forms. Then the simultaneous equations $f_n(x) = 0$, $g_n(x) = 0$ have an infinity of integral solutions with non-zero values of the variables when $n \geq 13$.

The author conjectures that the best possible result would be with 13 replaced by 10. He also gives conditions for the case when f has 11 variables and g has 12.

For $n = 5$ or 6 and with the additional assumption that there is one nontrivial integral solution of the simultaneous equations $f_n(x) = 0$, $g_n(x) = 0$ he shows that there are infinitely many solutions provided that for no real values of λ, μ is $\lambda f_n(x) + \mu g_n(x)$ either a singular form of rank $< n-1$ or: (a) if $n = 6$, a definite form of rank 5; (b) if $n = 5$, of rank 4 for rational ratios of λ, μ. *B. W. Jones* (Mayaguez, P.R.)

3379:

Vladimirov, V. S. On perfect forms in six variables. Mat. Sb. N.S. **44** (86) (1958), 263–272. (Russian)

Voronoi proved that a positive definite quadratic form

is extreme if and only if it is perfect and eutactic. [For definitions of these terms and simple proofs of Voronoi's theorem see M. Kneser, Canad. J. Math. 7 (1955), 145–149; MR 16, 1002; or E. S. Barnes, Proc. Cambridge Philos. Soc. 53 (1957), 537–539; MR 19, 120.] If $n \leq 5$, all classes of perfect forms in n variables are known and all of these are in fact extreme. Using an algorithm devised by Voronoi, Barnes has found all classes of perfect forms in six variables [cf. Philos. Trans. Roy. Soc. London, Ser. A 249 (1957), 461–506; MR 19, 251; and loc. cit.]. There are seven such classes, one of which is not extreme. The present author, working independently of Barnes (but a little later), obtains partial results in the same direction.

P. T. Bateman (Zbl 80, 264)

3380:

Cugiani, Marco. Forme quadratiche e cubiche binarie nei domini *P*-adici. Ist. Lombardo Accad. Sci. Lett. Rend. A 92 (1957/58), 307–320.

The author obtains necessary and sufficient conditions that a given binary quadratic form with p-adic integer coefficients should represent a given p-adic integer for p-adic integral values of the variables. He also obtains incomplete results for binary cubic forms. For earlier work of the author on special cases see Ann. Mat. Pura Appl. (4) 44 (1957), 1–22; Riv. Mat. Univ. Parma 8 (1957), 81–92 [MR 20 #1669, #7011].

J. W. S. Cassels (Cambridge, England)

3381:

Watson, G. L. Cubic forms representing arithmetic progressions. Proc. Cambridge Philos. Soc. 55 (1959), 270–273.

The author proves the following theorem. Let n_0 be an integer such that every cubic form with rational coefficients and at least n_0 variables represents zero nontrivially; then every non-degenerate cubic form with rational coefficients in at least n_0+2 integral variables represents an arithmetic progression. (The existence of n_0 between 10 and 32, has been shown by Lewis, Davenport and Birch.)

B. W. Jones (Mayaguez, P.R.)

3382:

Podsypanin, V. D. On representation of numbers by binary forms of fourth degree. Mat. Sb. N.S. 42 (84) (1957), 523–532. (Russian)

3383:

Carlitz, L. Some arithmetic properties of generalized Bernoulli numbers. Bull. Amer. Math. Soc. 65 (1959), 68–69.

H. W. Leopoldt [Abh. Math. Univ. Hamburg 22 (1958) 131–140; MR 19, 1161] has defined generalized Bernoulli numbers and polynomials as follows:

$$\sum_{n=0}^{\infty} B_\chi^n(u)\frac{t^n}{n!} = \sum_{r=1}^{f} \chi(r)\frac{te^{(r+u)t}}{e^{ft}-1}, \quad B_\chi^n = B_\chi^n(0),$$

where f is an integer ≥ 1 and χ is a primitive character (mod f). For $f=1$, B_χ^n is the ordinary Bernoulli number B_n. The main result of Leopoldt's paper is an analogue of the Staudt-Clausen theorem. The author states three theorems about B_χ^n (too long to be quoted here), one of which is a refinement of Leopoldt's result.

N. J. Fine (Princeton, N.J.)

3384:

Lambek, J.; and Moser, L. On some two way classifications of integers. Canad. Math. Bull. 2 (1959), 85–89.

It is shown that there is a unique way of splitting the non-negative integers into two classes in such a way that the sums of pairs of distinct integers (counting multiplicities) form the same set for both classes.

{Reviewer's note: While the method of proof, by a clever use of generating functions, uses the fact that one is dealing with integers, the result is clearly valid for arbitrary infinite cyclic semi-groups and, by an obvious extension, for direct products of such semi-groups. This includes the case of the natural numbers under multiplication which is treated separately in the paper.}

The case of finite sets is treated for initial segments of the non-negative integers. In general the problem of determining finite sets with the same set of sums of pairs would reduce to a study of a certain class of cyclotomic polynomials.

E. G. Straus (Los Angeles, Calif.)

3385:

Berg, Lothar. Über eine Differenzengleichung aus der Theorie der Partitionen. Wiss. Z. Univ. Rostock. Math.-Nat. Reihe 5 (1955/56), 269–278.

The author considers the difference equation $f_r(n) = f_{r-1}(n)+f_r(n-r)$ satisfied when $f_r(n)=p_r(n)$, the number of partitions of n into at most r parts, and obtains particular solutions of this equation of the form $f_r(n)=t^{-n}g_r(t)$. He then derives many known results for the functions $p_r(n)$ and $p(n)$ (the number of unrestricted partitions of n), the most striking of which is the inequality $p(n) < (\alpha/n)^{1/2}\exp 2(\alpha n)^{1/2}$, $\alpha=\pi^2/6$, derived in two or three lines.

The author also observes that a knowledge of the behavior of the function $\{(1-x)\cdots(1-x^r)\}^{-1}$ at $x=1$ would add greatly to our knowledge of the partition function and he gives recurrence formulas for the coefficients of the pole terms at $x=1$ of this function from which he works out the first few coefficients.

There are numerous other relationships given (well-known) and three short tables of $p_r(n)$.

M. Newman (Washington, D.C.)

3386:

Wright, E. M. The asymptotic behaviour of the generating functions of partitions of multi-partites. Quart. J. Math. Oxford Ser. (2) 10 (1959), 60–69.

Auluck [Proc. Cambridge Philos. Soc. 49 (1953), 72–83; MR 14, 726], Meinardus [Math. Ann. 132 (1956), 333–346; MR 18, 642] and E. M. Wright [Proc. London Math. Soc. (3) 7 (1957), 150–160; MR 19, 16] have found asymptotic formulae for the number of partitions of a multi-partite number with large components (a multi-partite number is a j-dimensional vector with positive integral components). A simple example of a generating function one studies is

$$f(x_1, \cdots, x_j) = \prod_{h_1,\cdots,h_j=1}^{\infty} (1-\exp(-h_1x_1-\cdots-h_jx_j))^{-1}$$
$$= \sum_{n_1,\cdots,n_j=0}^{\infty} p(n_1,\cdots,n_j)\exp(-n_1x_1-\cdots-n_jx_j),$$

where re $(x_l)>0$ for every l and $p(0,\cdots,0)=1$. The author is concerned with approximating to f for small x_l, and his results are sharper than those in the previous work mentioned. The author remarks that one cannot expect

the same accuracy in an approximation to f as has been achieved in the case $j=1$ [see G. H. Hardy and S. Ramanujan, Proc. London Math. Soc. (2) **17** (1918), 75–115].

S. Chowla (Boulder, Colo.)

3387:

Erdös, P. Some remarks on prime factors of integers. Canad. J. Math. **11** (1959), 161–167.

The following theorems are proved. I. Let $\varepsilon_p>0$, $\delta_p=\min(\varepsilon_p, 1)$. Then the divergence of $\sum_p \delta_p/p$ is a necessary and sufficient condition that all integers except for a set of density zero should have two prime factors p and q satisfying the inequality $p<q<p^{1+\varepsilon_p}$. II. The density of integers n which have two prime factors p and q such that $p<q<p^{1+c/\log\log n}$ equals $1-e^{-c}$.

In the proof of I, Turán's method of the second moment is used; in the proof of II, a sieve argument occurs.

W. J. LeVeque (Ann Arbor, Mich.)

3388:

Yin, Wen-lin. Piltz's divisor problem for $k=3$. Sci. Record (N.S.) **3** (1959), 169–173.

Let θ be the lower bound of α such that $\sum_{n\leq x} d_3(n)=xP(\log x)+O(x^\alpha)$, where P is a certain quadratic. Improving on a recent result of Yüh Ming-I [same Record **2** (1958), 326–328; MR **21** #35], the author gives the result $\theta\leq 25/52$, and indicates the main lemmas and steps in the proof. Other methods are stated to yield the even better results $\theta\leq 10/21$, $\theta\leq 8/17$.

F. V. Atkinson (Canberra)

3389:

Wirsing, Eduard. Bemerkung zu der Arbeit über vollkommene Zahlen. Math. Ann. **137** (1959), 316–318.

Let $\delta(n)=\sum_{d|n} d$. For every rational number k let $V_k(x)$ denote the number of integers $v\leq x$ such that $\delta(v)=kv$. Hornfeck and Wirsing proved $V_k(x)=O(x^\varepsilon)$ [Math. Ann. **133** (1957), 431–438; MR **19**, 837]. In the present paper this result is improved: There are constants $c>0$ and x_0 such that $V_k(x)\leq \exp(c \ln x/\ln\ln x)$, for all $x\geq x_0$ and all k. The proof is based on a decomposition of the numbers v according to the size of their prime factors.

P. Scherk (Toronto, Ont.)

3390:

Buschman, R. G. Asymptotic expressions for $\sum n^a f(n) \log^r n$. Pacific J. Math. **9** (1959), 9–12.

There exists a class of Tauberian theorems which assert that if $F(t)$ is a non-negative, non-decreasing function in $0\leq t<\infty$ and various regularity conditions are satisfied, an asymptotic expression for the Laplace transform $\mathscr{L}\{F\}$ of F implies a related asymptotic expression for $F(t)$ [see H. Delange, Ann. Sci. École Norm. Sup. (3) **71** (1954), 213–242; MR **16**, 921]. Such theorems have a number of well-known applications in number theory. The author applies this method to functions $F(t)=\sum_{1\leq n\leq e^t} n^a f(n)\log^r n$ where f is a non-negative number-theoretic function, and lists a large number of results, several of them proved in some detail.

H. Halberstam (London)

3391:

Haselgrove, C. B. A disproof of a conjecture of Pólya. Mathematika **5** (1958), 141–145.

The conjecture of Pólya asserts that for every $x\geq 2$ there are at least as many numbers $\leq x$ having an odd number of prime factors as there are numbers with an even number of prime factors. More precisely, if we adopt Liouville's function

$$\lambda(n) = (-1)^{a_1+\cdots+a_r} \quad (n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}),$$

then $L(x)=\sum_{n\leq x}\lambda(n)$ is never positive for $x\geq 2$. This conjecture had been verified as far as 800000 by R. S. Lehman and W. G. Spohn. The author shows that the conjecture is false. Furthermore there is a good reason to believe $L(x)>0$ for x near $e^{831.847}=1.845\cdot 10^{361}$. In a note added in proof, the author announces that he has disproved also the conjecture of Turán, namely that $L_1(x)=\sum_{n\leq x}\lambda(n)/n$ is never negative.

The method is based on results of A. Ingham [Amer. J. Math. **64** (1942), 313–319; MR **3**, 271] involving the two functions

$$A(u) = e^{-u/2}L(e^u),$$

$$A_T^*(u) = \alpha_0 + 2\,\mathrm{Re}\,\{\sum (1-\gamma_n/T)\alpha_n \exp(iu\gamma_n)\},$$

where the sum extends over those values of n for which $0<\gamma_n<T$ and γ_n is the imaginary part of the nth complex zero ρ_n of the Riemann zeta-function. Also

$$\alpha_0 = 1/\zeta(1/2), \quad \alpha_n = \zeta(2\rho_n)/(\rho_n\zeta'(\rho_n)).$$

The conjecture of Pólya implies the Riemann Hypothesis and so to disprove the former we may assume the latter. Ingham showed that on the assumption of the Riemann Hypothesis

$$A_T^*(u) \leq \limsup A_T^*(u) \leq \limsup A(u),$$

and so to disprove the Pólya conjecture it suffices to discover T and u for which $A_T^*(u)>0$. Such a pair is $T=1000$, $u=831.847$ for which $A_T^*(u)=.00495$. This discovery was based on a calculation of the first 1500 zeros ρ_n and the corresponding values α_n. The latter complex constants and γ_n are tabulated on p. 141 for $n=1\ (1)\ 50$ to 80. There is also a short table of $A^*_{1000}(u)$ for $u=831.800\ (.001)\ 831.859$. {There is obviously something wrong at $u=831.857$. Presumably the value should be -0.06321, not -1.56321.} To show that $A_T^*(u)$ and $A(u)$ are not very far apart there is a graph of both functions (for $T=200$) for $x=e^u=44400\ (200)\ 51600$, showing the near miss at $x=48512$ where $L(x)=-2$.

The calculations were made independently on EDSAC I and Manchester University's Mark I. They have since been confirmed by R. S. Lehman using an IBM 701. The author believes that Merten's conjecture $\sum_{n\leq x}\mu(n)<x^{1/2}$ may be disproved by the use of very much faster machines.

D. H. Lehmer (Berkeley, Calif.)

3392:

Pólya, G. Heuristic reasoning in the theory of numbers. Amer. Math. Monthly **66** (1959), 375–384.

Let d be a fixed even integer, and $x>0$. It was conjectured by Hardy and Littlewood [Acta Math. **44** (1923), 1–70] that the number of primes $p\leq x$ such that $p+d$ is again a prime, is asymptotically $Cx(\log x)^{-2}$, where

$$C = 2\prod_{p\geq 3}(1-(p-1)^{-2}) \prod_{p|d,\, p \text{ odd}} ((p-1)/(p-2)).$$

The author shows how this formula can be guessed by heuristic reasoning on a probability basis, and he draws attention to the fact that his type of reasoning is very

similar to the physicist's way of thinking. The paper includes numerical material obtained by Prof. and Mrs. D. H. Lehmer, covering $x=3.10^7$, $k=2(2)70$.

N. G. de Bruijn (Amsterdam)

3393:

Kesten, Harry. On a series of cosecants. II. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 110–119.

[For part I, see M. Kac and R. Salem, same Proc. **60** (1957), 265–267; MR **19**, 646.] Let x be a random variable, uniformly distributed on $[0, 1]$. It is shown that $\sum_0^\infty c_k \csc 2\pi 2^k x$ converges or diverges with probability 1, according as $\sum_0^\infty |c_k|$ converges or diverges. Also, if $a_n \to \infty$ with n, then $a_n^{-1} \sum_0^{n-1} \csc 2\pi 2^k x \to 0$ in probability, i.e.,

$$\lim_{n\to\infty} \operatorname{Prob}\left\{ \left| a_n^{-1} \sum_0^{n-1} \csc 2\pi 2^k x \right| > \varepsilon \right\} = 0$$

for every $\varepsilon > 0$. The latter theorem depends strongly on special properties of the sine function, since it is also shown that $n^{-1/2} \sum_0^{n-1} \varphi(2^k x)$ is asymptotically normally distributed, where $(\varphi(t))^{-1}$ is a certain linear approximation to $\sin 2\pi t$.

W. J. LeVeque (Ann Arbor, Mich.)

3394:

Reichardt, Hans. Eine Bemerkung zur Theorie des Jacobischen Symbols. Math. Nachr. **19** (1958), 171–175.

The author considers a definition of the Jacobi nth order residue symbol in an arbitrary algebraic field containing ζ, an nth root of unity, by using Gauss' lemma modified for a partition of relatively prime residues into n categories mutually incongruent through factors of ζ^k. He proves the multiplicative properties for the symbol directly, without reciprocity, and notes a variant of Gauss' famous Third Proof. {For such a highly documentable subject, there is an unfortunate dearth of references since only one is cited: A. Scholz, *Einführung in die Zahlentheorie* [2nd. ed., de Gruyter, Berlin, 1955; MR **17**, 127].

H. Cohn (Tucson, Ariz.)

3395:

Gundlach, Karl-Bernhard. Dirichletsche Reihen zur Hilbertschen Modulgruppe. Math. Ann. **135** (1958), 294–314.

The author continues his earlier development [Math. Z. **64** (1956), 339–352; MR **18**, 195] and establishes a functional equation for his Hecke-Eisenstein series in a totally real field of degree n:

$$G_{-r}^*(\tau, s, \mathfrak{a}, \Gamma(1, \mathfrak{c})) = \sum N(m_1^*\tau + m_2^*)^{-r} |N(m_1^*\tau + m_2^*)|^{-s}$$

for m_1^*, m_2^* determined by the ideals $\mathfrak{a}$ and $\mathfrak{c}$ in a complicated manner described in the earlier paper. By a direct transformation of Fourier series, he relates the series to

$$G_{-r}^*(\tau, 2(1-r)-s, \mathfrak{q}, \Gamma(1, \mathfrak{c}))$$

for $\mathfrak{q} \sim \mathfrak{c}\mathfrak{a}^{-1}\mathfrak{b}^{-1}$ ($\mathfrak{b}$ = different). The natural restriction is $\operatorname{Re} s + r > 2$ but the author finds the residue (for $r = 0$) at $s = 2$, and effects analytic continuation. The main significance of the result seems to be a proof that for the Fourier coefficients $a(\mu+\kappa) = ON(\mu+\kappa)^{r/2-1/(4n+1)}$. Here μ belongs to an appropriate module (and κ is taken from a finite set of residues as required to cancel out the multipliers introduced by the translations of the modular group). The determination of κ is described in the author's work [Math. Ann. **117** (1940), 538–540]. The asymptotic result is not as strong as the author's result in the work last cited: $a(\mu) = ON(\mu)^{r/2-1/4+\varepsilon}$.

H. Cohn (Tucson, Ariz.)

3396:

Meyer, C. Über einige Anwendungen Dedekindscher Summen. J. Reine Angew. Math. **198** (1957), 143–203.

The applications of Dedekind sums referred to in the title of this paper concern the problem of the determination of the class number of algebraic number fields K such that the least normal closure of K is an abelian extension of some quadratic field Ω. This problem was completely solved by the author [*Die Berechnung der Klassenzahl Abelscher Körper über quadratischen Zahlkörpern*, Akademie-Verlag, Berlin, 1957; MR **19**, 531; cf. also Hasse, Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10** (1951), 84–95; MR **14**, 140]. The solution requires the evaluation of certain L-series $L(s, \mathfrak{k})$ for $s = 1$, belonging either to ring classes $\mathfrak{k}$ modulo fp_∞ or ray classes $\mathfrak{k}$ modulo $\mathfrak{f}p_\infty$ in a real quadratic number field Ω with discriminant $d > 0$, where f is a rational integer, $\mathfrak{f}$ an integral divisor of Ω and p_∞ is the infinite rational prime. In the case of ring classes the result is

$$\text{(A)} \quad L(1, \kappa) = \frac{2\pi}{f\sqrt{d}} \Im\left(\log \frac{\sqrt[24]{\Delta(a\omega_1 + b\omega_2,\ c\omega_1 + d\omega_2)}}{\sqrt[24]{\Delta(\omega_1, \omega_2)}}\right),$$

where

$$\sqrt[24]{\Delta(\omega_1, \omega_2)} = \left(\frac{2\pi}{\omega_2}\right)^{1/2} \eta(\tau)$$

is the discriminant in the theory of elliptic functions and

$$\eta(\tau) = q^{1/24} \prod_{m=1}^\infty (1-q^m) \quad \text{with} \quad \tau = \frac{\omega_1}{\omega_2}, \quad \Im(\tau) > 0,$$

is Dedekind's function. The rational integers a, b, c, d with $ad - bc = 1$, $c \neq 0$, depend on the choice of a divisor $\mathfrak{c}$ in $\mathfrak{k}$ and on the choice of a basis of the ideal $(\mathfrak{c})_f$ consisting of all multiples of this divisor in the ring of the numbers of Ω whose denominators are prime to f. The value of $L(1, \mathfrak{k})$ must be independent of these choices. The author poses the problem of finding direct elementary proofs for these (and some other) invariance properties [this problem was already mentioned by Hecke in the case of a totally imaginary biquadratic number field in his paper in the Göttinger Nachr. **1921**, 1–23; *Mathematische Werke*, Vandenhoeck and Ruprecht, Göttingen, 1959; MR **21** #3303; pp. 290–312, particularly p. 311]. Since (A) can also be written as

$$L(1, \mathfrak{k}) = -\frac{2\pi^2}{f\sqrt{d}} \operatorname{sgn} \delta(\mathfrak{c})\left(-\frac{a+d}{12c} + \frac{1}{4}\operatorname{sgn} c + \operatorname{sgn} c \cdot s(a, c)\right),$$

where $\delta(\mathfrak{c})$ is the discriminant of a basis for the ideal $(\mathfrak{c})_f$ and $s(a, c)$ is a Dedekind sum, the problem is essentially one about Dedekind sums. In fact its solution follows from the reciprocity formula for these sums. In the first part of his paper the author gives a systematic theory of Dedekind sums. Most of this account is, however, well known from the work of Rademacher and others.—In the case of ray classes considered in the second part of the paper the analogous problem is more complicated. Klein's functions $\sigma_{gh}(\omega_1, \omega_2)$ (essentially "Teilwerte" of Weierstrass's σ-function) must now be considered instead of $\eta(\tau)$. Especially it is necessary to study the behavior of log σ_{gh} under

modular substitutions belonging to the principal congruence subgroup $\Gamma(f)$ of the modular group. This led the author already in his book cited above to the introduction of generalized Dedekind sums defined by

$$S_{gh}(a, c) = \sum_{\mu \bmod c} P_1\left(\frac{a\mu}{c} + \frac{ag+ch}{fc}\right) P_1\left(\frac{\mu}{c} + \frac{g}{fc}\right),$$

where μ runs through a complete set of residues modulo c, and $P_1(x) = x - [x] - \frac{1}{2}$. In the second part of the paper the author gives an account of the theory of these sums. They can be reduced to ordinary Dedekind sums and some other sums introduced by Eisenstein and Stern.

H. D. Kloosterman (Leiden)

3397:

Dieter, Ulrich. Das Verhalten der Kleinschen Funktionen log $\sigma_{g,h}(\omega_1, \omega_2)$ gegenüber Modultransformationen und verallgemeinerte Dedekindsche Summen. J. Reine Angew. Math. **201** (1959), 37–70.

The behavior of the logarithm of F. Klein's function $\sigma_{gh}(\omega_1, \omega_2)$ (which is essentially $\sigma\left(\frac{g\omega_1 + h\omega_2}{f}, \omega_1, \omega_2\right)$, where $\sigma(u, \omega_1, \omega_2)$ is Weierstrass's σ-function; g, h, f integers) under modular substitutions in ω_1, ω_2, belonging to the principal congruence subgroup $\Gamma(f)$ of level ("Stufe") f has been studied by C. Meyer [see review above]. The author now studies the behavior of the functions σ_{gh} under arbitrary modular substitutions. The method used is that of Rademacher in the case of Dedekind's function $\eta(\tau)$ [same J. **167** (1931), 312–336].

H. D. Kloosterman (Leiden)

3398:

Fenna, D. Simultaneous Diophantine approximation to series. J. London Math. Soc. **34** (1959), 173–176.

Let k denote an arbitrary nontrivial field, z an indeterminate, and K the field of formal Laurent series

$$x = \alpha_d z^d + \alpha_{d-1} z^{d-1} + \cdots$$

with coefficients in k. For a fixed real number $\kappa > 1$, define a valuation on K by $|0| = 0$ and $|x| = \kappa^d$ if α_d is the leading nonzero coefficient in $x \neq 0$. Let $c(n)$ be the supremum of the numbers c such that for all $t_1, \cdots, t_n \in K$ which are not all rational functions of z, there are infinitely many sets of polynomials $b_0, b_1, \cdots, b_n \in k[z]$ satisfying

$$|b_0(b_0 t_i - b_i)^n| \leq \kappa^{-c} \quad (i = 1, 2, \cdots, n).$$

It is shown that $c(n) = n$.

W. J. LeVeque (Ann Arbor, Mich.)

3399:

Negoescu, N. Inégalités pour certaines fractions continues multiples. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **4** (1958), 1–9. (Romanian. Russian and French summaries)

Let $[a_0, a_1, \cdots, a_v, \cdots]$ represent the continued fraction expansion of an irrational number θ, and $[a_0, a_1, \cdots, a_v]$ be the approximant p_v/q_v of order v. Let

$$[a_{v+1}, a_{v+2}, \cdots] + [0, a_v, a_{v-1}, \cdots, a_1]$$

be the double continued fraction

$$[a_1, \cdots, a_v, [a_{v+1}], a_{v+2}, \cdots],$$

introduced by R. Robinson [Bull. Amer. Math. Soc. **53** (1947), 351–361; **54** (1948), 693–705; MR **8**, 566; **10**, 235]. This concept is generalized to that of a multiple continued fraction. Let

$$[A_0; a_{v,1}; a_{v,2}; \cdots; a_{v,m}]$$

be the m-tuple continued fraction which designates the sum

$$\sum_{i=1}^{m} [a_0, a_1, a_2, \cdots, a_{v-1}, a_{v,i}]$$

of m regular continued fractions. A number of inequalities are proved here on multiple continued fractions, based on the author's previous work [An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) **3** (1957), 11–17; Gaz. Mat. Fiz. Ser. A (N.S.) **10** (**63**) (1958), 482–491; MR **20** #4544, #5988]. The principal theorem is as follows: Let a_0, $a_1, \cdots, a_{v-1}$ be integers $\geq n$, m and s integers ≥ 2, γ_i and δ_i real numbers $> n$. Let A_0 be the block $a_0, a_1, \cdots, a_{v-1}$, and v an even number. Then

(a) $[A_0; s-1, n, \gamma_1; \cdots; s-1, n, \gamma_{m-1}; s, n, \gamma_m] < [A_0; s, n, \delta_1; \cdots; s, n, \delta_m]$,

(b) $[A_0; s+1, n, \gamma_1; \cdots; s+1, n, \gamma_{m-1}; s, n, \gamma_m] > [A_0; s, n, \delta_1; \cdots; s, n, \delta_m]$,

where

$$\delta_i < (m-1)(n^2+1)^2 s + (3-m)n(n^2+n+1) \quad (i = 1, 2, \cdots, m).$$

E. Frank (Chicago, Ill.)

3400:

Veldkamp, G. R. Some remarks on continued fractions. Nieuw Tijdschr. Wisk. **46** (1958/59), 173–182. (Dutch)

Expository article [the material can be found in O. Perron, *Die Lehre von den Kettenbrüchen*, vol. I, 3te Aufl., Teubner, Stuttgart, 1954; MR **16**, 239; pp. 36–42].

N. G. de Bruijn (Amsterdam)

3401:

Walfisz, Arnold. Gitterpunkte in mehrdimensionalen Kugeln. Jber. Deutsch. Math. Verein. **61** (1958), Abt. 1, 11–31.

The author lists the important results in the historical development of the problem of finding the number of lattice points in a many-dimensional sphere. He states these as theorems, thirty-seven in all, without complete proofs, but follows each theorem with brief comments and a reference to where the complete proof can be found.

W. H. Simons (Vancouver, B.C.)

3402:

Rédei, L. Neuer Beweis eines Satzes von Delone über ebene Punktgitter. J. London Math. Soc. **34** (1959), 205–207.

The author gives a neat proof of the theorem of Delone [Izv. Akad. Nauk SSSR. Ser. Mat. **11** (1947), 505–538; MR **9**, 334] that every 2-dimension lattice with no points on the coordinate axes contains a 'split parallelogram'.

J. W. S. Cassels (Cambridge, England)

3403:

Woods, A. C. A counter-example in the geometry of numbers. Quart. J. Math. Oxford Ser. (2) **10** (1959), 46–47.

The example concerns the critical lattices of a plane convex body K and those of a body K^* obtained by rotating K.

C. G. Lekkerkerker (Amsterdam)

3404:

Świerczkowski, S. On successive settings of an arc on the circumference of a circle. Fund. Math. **46** (1959), 187–189.

Let N be a positive integer and C a directed circle whose circumference is not an integer $\leq N$. Let P be an arbitrary point on C and let p_x $(x=0, 1, \cdots, N)$ denote the point arrived at on going arc length x from P in the positive direction. Consider now the points p_x in the order in which they are situated on C, let p_{a_r} be the point which immediately follows p_0 and p_{a_k} the one immediately followed by p_0. Proving a conjecture of H. Steinhaus the author shows that if p_y follows immediately p_x then $y-x$ takes one of the values a_r, $-a_k$, a_r-a_k. It is stated that two other (unpublished) proofs are due to P. Szüsz and P. Erdös and V. S. Turán. *A. Dvoretzky* (Jerusalem)

FIELDS

See also 3356, 3394, 3396, 3397, 3398, 3465.

3405:

Roquette, Peter. Einheiten und Divisorklassen in endlich erzeugbaren Körpern. Jber. Deutsch. Math. Verein. **60** (1957), Abt. 1, 1–21.

Let K be a field. A prime divisor $\mathfrak{p}$ of K means a class of equivalent discrete valuations of rank 1. Let M be a set of prime divisors of K such that (1) for every element $a \in K^*$ (the multiplicative group of K) there exist only a finite number of $\mathfrak{p} \in M$ with $w_{\mathfrak{p}}(a) \neq 0$. The divisor group D_M is defined to be the free abelian group with free generators $\mathfrak{p} \in M$. Then the natural exact sequence

$$1 \rightarrow E_M \rightarrow K^* \rightarrow D_M \rightarrow C_M \rightarrow 1$$

holds with the unit group E_M and the divisor class group C_M. The problem is to show that under certain conditions E_M and C_M are finitely generated.

The "Stufe" of a finitely generated field K means (the transcendental degree of K)+1 if the characteristic of K is zero, and the transcendental degree otherwise. It is denoted by St (K). For a subset x of K, St $(x)=K$ means that K is finite algebraic over (x). Then M_x means the set of all prime divisors $\mathfrak{p}$ of K such that (a) every element of x is integral for $\mathfrak{p}$ and (b) St $(x \bmod \mathfrak{p})=$ St $(K \bmod \mathfrak{p})=$ St $(K)-1$. It is proved that M_x has the above property (1) in case St $(x)=K$. In general a set M of prime divisors of K is called finitely generated if there exists a set $x \subset K$ such that St $(x)=K$, $M_x \subset M$ and $M-M_x$ is finite.

The fundamental results are the following two theorems. Unit theorem: Let K be finitely generated; then the unit group E_M with respect to a finitely generated set M of prime divisors is itself finitely generated. Class theorem: Under the same assumptions the divisor class group C_M is also finitely generated. These theorems are proved by mathematical induction on the Stufe of K. The unit theorem follows from well-known results in the case of Stufe 1. For the proof of the class theorem the deep results of Weil and Néron are used. *Y. Kawada* (Tokyo)

3406:

Krull, Wolfgang. Über einen Existenzsatz der Bewertungstheorie. Abh. Math. Sem. Univ. Hamburg **23** (1959), 29–35.

Hasse proved in 1925 [Math. Ann. **95** (1925), 229–238] that if B_i $(i=1, 2, \cdots, t)$ is a discrete valuation of a finite algebraic number field F with residue field k_i, if $K_{i1}, K_{i2}, \cdots, K_{is_i}$ are algebraic extensions of k_i of degrees $f_{i1}, f_{i2}, \cdots, f_{is_i}$ respectively, and if $e_{i1}, e_{i2}, \cdots, e_{is_i}$ are positive integers such that $\sum_j f_{ij}e_{ij}=n$ $(i=1, 2, \cdots, t)$, then there is an algebraic extension G of F of degree n such that B_i admits exactly s_i extensions $B_{i1}, B_{i2}, \cdots, B_{is_i}$ to G, the residue field of B_{ij} is isomorphic to K_{ij}, and e_{ij} and f_{ij} are the ramification and inertial indices of B_{ij} relative to B_i. The present paper gives three theorems which extend this result to the case where F is an arbitrary field. In each of these the additional requirement that K_{ij} be a simple extension of k_i is imposed, and the assertion that G is separable over F is added to the conclusion, which is otherwise unchanged. The first of these theorems imposes the additional hypothesis that $s_i=1$ for at least one value of i, and the third requires instead that F admit at least one discrete valuation B distinct from each of $B_1, B_2, \cdots, B_t$. If D is the intersection of the valuation rings of $B_1, B_2, \cdots, B_t$ the second theorem has as its added hypothesis the condition that the coefficients of any monic polynomial in $D[X]$ can be approximated uniformly by the coefficients of irreducible separable polynomials in each of the topologies defined by the powers of the maximal ideals of D.
H. T. Muhly (Iowa City, Iowa)

3407:

Lamprecht, Erich. Zur Klassifikation von Differentialen in Körpern von Primzahlcharakteristik. I. Math. Nachr. **19** (1958), 353–374.

Let A be an algebraic function field of one variable with constant field K, let there be given a real discrete valuation of A that is nontrivial on K, let $\bar{A}$, $\bar{K}$ be the residue fields, and assume certain regularity conditions. There is then a natural map of divisors of A into those of $\bar{A}$, and similarly for differentials. The classification of differentials into first kind, second kind, exact, is not disturbed by this latter map in the equal characteristic case, but if A, $\bar{A}$ have characteristics 0, $p>0$ respectively, then exact differentials of A go into pseudoexact differentials of $\bar{A}$, these latter being defined as linear combinations with coefficients in $\bar{K}$ of exact differentials and ones of the form $\bar{t}^{p^n-1}d\bar{t}$. Pseudoexact differentials are without residue and form an infinite-dimensional vector space modulo exact differentials. *M. Rosenlicht* (Berkeley, Calif.)

3408:

Eichler, Martin. Ein Satz über Linearformen in Polynombereichen. Arch. Math. **10** (1959), 81–84.

Let k be an arbitrary field, x an indeterminate and $k_\infty(x)$ the field of formal power series in x^{-1} over k. Let $d(a)$ denote the degree of a function a in $k_\infty(x)$ or in $k[x]$ (thus, e.g., $d(x^{-3})=-3$). The author proves the following theorem: Let $M=(m_{ij})$ be any $n \times n$ non-singular matrix over $k_\infty(x)$ and let $\tilde{M}=(\tilde{m}_{ij})$ be the contragredient of M: $\tilde{M}={}^tM^{-1}$. For any integers $\gamma_1, \cdots, \gamma_n$, let $l(M, \gamma)$ and $l(\tilde{M}, \tilde{\gamma})$ denote the dimensions of the vector spaces over k formed by all $(x_1, \cdots, x_n)$ and $(\tilde{x}_1, \cdots, \tilde{x}_n)$, respectively, satisfying

$$d\Big(\sum_i x_i m_{ij}\Big) \leq \gamma_j, \qquad x_i \in k[x],$$

$$d\Big(\sum_i \tilde{x}_i \tilde{m}_{ij}\Big) \leq \tilde{\gamma}_j = -2-\gamma_j, \quad \tilde{x}_i \in k[x], \quad i, j = 1, \cdots, n.$$

Then $l(M, \gamma) = l(\bar{M}, \tilde{\gamma}) - d(|M|) + \sum_i \gamma_i + n$. It is also noticed that the theorem of Riemann-Roch is a consequence of this equality. *K. Iwasawa* (Cambridge, Mass.)

ALGEBRAIC GEOMETRY

See also 3408, 3466.

3409:

★Severi, Francesco. Geometria dei sistemi algebrici sopra una superficie e sopra una varietà algebrica. Vols. 2, 3. Volumi secondo e terzo in continuazione del volume primo dello stesso autore che porta il titolo: Serie, sistemi d'equivalenza e corrispondenze algebriche sulle varietà algebriche. Edizioni Cremonese, Rome. Vol. 2. 1958, iv+464 pp. 4500 Lire; Vol. 3. 1959, viii+447 pp. 4800 Lire.

The first volume of this major work (1942) was briefly noticed in MR **10**, 206. Vol. 2 contains the theory of continuous systems of curves and algebraic subvarieties which lie on an algebraic variety, and also the beginning of the theory of the simple and multiple integrals attached to a surface and an algebraic variety. This theory is continued in the third volume.

The methods used throughout are acknowledged to be those of classical Italian geometry, but the author rightly stresses that these books will be useful, not only for the abundant material contained therein, but also for the extensions which can be expected when results contained here are developed in the abstract domain.

Some chapters are followed by "Notizie e riflessioni storico-critiche", and the many admirers of Prof. Severi will read these with especial pleasure, even when some of the shafts with which every page bristles are aimed at themselves. To judge between the author's many claims to priority in the discovery of fundamental theorems in algebraic geometry and those of other writers would demand a Solomon able to match Prof. Severi both in mathematical ability and in mathematical scholarship, and such people are very rare indeed. The reviewer is only too conscious that he is not one of the elect, and must say nothing.

It should be noted, however, that although the author is modest in describing the scope of the two volumes under discussion, vol. 3 contains a whole chapter on Hodge's theorem on the periods of integrals of the first kind, and the following chapters, which include an appendix by Prof. Marchionna on the Riemann-Roch theorem, mention all the most recent work in this field.

The two volumes are beautifully produced, and will have an important place on every algebraic geometer's bookshelves. It is to be hoped that the author, who has contributed so much to his beloved science, will not, as he indicates in the preface to vol. 3, cease from writing merely because he has reached his eightieth year. The delightful quotation from Alfred de Musset: "mon verre est petit, mais je bois dans mon verre" in the same preface seems to indicate, very fortunately, that the mere weight of years will not curb this indomitable spirit.

D. Pedoe (Singapore)

3410:

Godeaux Lucien. Sur la dégénérescence d'une sextique plane. Mathesis **68** (1959), 5–7.

"Nous démontrons le théorème suivant: Une sextique plane possédant six tacnodes non situés sur une conique, dégénère en trois coniques deux à deux bitangentes."

Résumé de l'auteur

3411:

Buquet, A. Sur la recherche directe de l'indépendance de plusieurs points rationnels donnés d'une cubique. Mathesis **68** (1959), 24–37.

A set of p rational points $A_1, A_2, \cdots, A_p$ on a plane cubic curve are "independent" if it is not possible to find $q\,(<p)$ rational points $B_1, B_2, \cdots, B_q$ on the curve from which one can find the A_j by repeated operation of alignment. In a previous paper [Mathesis **62** (1953), 281–289; MR **15**, 400] the author has considered the special case of cubics of the form $x^3+dx+e=z^2$; this investigation is now extended to arbitrary cubics. The problem is reduced to that of determining whether certain equations of the fourth degree (in a single unknown) have rational solutions.

L. Carlitz (Durham, N.C.)

3412:

Green, H. G.; and Prior, L. E. Some involution and incidence properties of the twisted cubic. Ann. Soc. Sci. Bruxelles. Sér. I **73** (1959), 235–244.

The projection on a plane of a twisted cubic with seven base points from one of these points is a conic on which the projections of the other six base points are vertices of inscribed hexagons. These six points on the conic taken in any order define sixty Pascal lines. Properties of certain lines in three dimensions associated with the twisted cubic are derived from the properties of the two-dimensional Pascal configurations. The method of projection and Pomey's extended concept of the Fregier point are used.

The authors show that any chord of the twisted cubic through two Fregier points is a common generator of three quadric surfaces, each containing the cubic and two additional chords each of which joins a pair of base points. The existence of this common generator has been noted heretofore, but it is explicitly defined here for the first time. *T. R. Hollcroft* (Aurora, N.Y.)

3413:

Godeaux, Lucien. Sur les surfaces de genres géométrique et arithmétique nuls possédant un faisceau de courbes bicanoniques irréductibles. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 738–749; addition, 942–944.

In two former papers [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (6) **14** (1931), pp. 479–481; Acad. Roy. Belg. Bull. Cl. Sci. (5) **18** (1932), 26–37] the author showed that if a non-singular quintic surface F in S_3 (regular-canonical model, with $p_g=p_a=4$, $p^{(1)}=6$) has on it a cyclic involution of order 5, generated by a collineation of period 5 in S_3, with four fixed points none of which is on F, then a model F' of the involution has $p_a=p_g=0$, $p^{(1)}=P_2=2$, the bicanonical system being a pencil of irreducible curves. The present paper is devoted to showing that if a surface F' has $p_a=p_g=0$, $p^{(1)}=P_2=2$, the bicanonical system being a pencil of irreducible curves, whose four base points are distinct, the surface is in fact that obtained above; this is proved by constructing the tricanonical model, a septimic surface in F_3 having a skew quadrilateral of tacnodal lines.

The addition simplifies part of the main proof.

P. Du Val (London)

3414:

Godeaux, L. Sur les involutions cycliques privées de points unis appartenant à une surface algébrique. Colloq. Math. 6 (1958), 5–12.

F is a surface of genera p_g, p_a and linear genus π. If F has on it a cyclic involution I of order p, induced by a birational self transformation of period p without fixed points, base points, or exceptional curves, an image F' of I has genera p'_g, p'_a, π' satisfying

$$p_a+1 = p(p'_a+1), \quad \pi-1 = p(\pi'-1),$$
$$p_g-p_a \geq p'_g-p'_a.$$

If $p<p_a+1$, or if $p=p_a+1$ and $p'_g>p'_a=0$, there is an effective canonical system $|K_0|$ on F', and $p-1$ systems $|K_i|$ $(i=1, \cdots, p-1)$ of dimension p'_a which appear to differ from $|K_0|$ by zero divisors which form a cyclic group of order p, though this does not seem to be stated. $|K_0|, \cdots, |K_{p-1}|$ are the images of subsystems compounded with I of the canonical system on F. The bicanonical system and the adjoints $|K'_i|$ of $|K_i|$ $(i=1, \cdots, p-1)$ are similarly the images of subsystems compounded with I of the bicanonical system on F.

If $p=p_a+1$ and F' is regular, $|K_0|$ is of course merely virtual, but the other $2p-1$ systems mentioned above are effective. *P. Du Val* (London)

3415:

Andreotti, Aldo; e Salmon, Paolo. Anelli con unica decomponibilità in fattori primi ed un problema di intersezioni complete. Monatsh. Math. 61 (1957), 97–142.

Let K be a commutative field; let $\mathfrak{p}$ be a prime ideal of the graded ring $K[x_0, \cdots, x_n]$, which defines the algebraic variety V_k of dimension $k\geq 2$, and let $\mathcal{I}=K[x_0, \cdots, x_n]/\mathfrak{p}$. The problem here considered is the following: to recognize when V_k contains only subvarieties of the greatest dimension $k-1$, which are complete intersections with a hypersurface of its space; we shall say that V_k contains only complete intersections. In this case every pure $(k-1)$-dimensional ideal in $\mathcal{I}$ is principal and vice versa. The main theorem is the following. (A): A necessary and sufficient condition that every pure $(k-1)$-dimensional ideal in $\mathcal{I}$ ($\mathcal{I}$ normal) may be principal is that $\mathcal{I}$ should be Z.P.E.

The second part of this work is concerned with the applications of (A). As an immediate consequence some theorems by Bertini and Klein are obtained; it is demonstrated that on the general curve of genus p, $3\leq p\leq 5$, every complete linear series, rationally determined, is a multiple of the canonical series; it is found again that Grassmann varieties contain only complete intersections. In a second group of applications there are, as particular cases, two theorems by Severi and by M. Noether. The more comprehensive theorems here demonstrated are the following: (a) If the irreducible hypersurface $f(x)=0$ of the projective space P_n, $n\geq 4$, contains an incomplete intersection, it is singular and has a variety of dimension $\geq n-4$ of double points (i.e., points where, over a suitable extension of the ground field, all the partial derivatives of $f(x)$ are zero). (b) If the irreducible nonsingular surface $f(x)=0$ of the projective space P_3 contains an incomplete intersection, then it is possible to write $f(x)$ in the form of a homogeneous determinant of order $s\geq 2$. The theorem (a) contains, in particular, one of Severi. From (b) is derived the theorem of M. Noether, which says that a generic surface belonging to P_3, of order $l\geq 4$, contains only complete intersections. To obtain this second result it is necessary to demonstrate that the surface contains no incomplete perfect intersections. This is obtained with a very elaborate demonstration for which it is necessary to suppose the ground field of characteristic zero. A further application of (A) is the theorem by Appell and Humbert concerning the intermediate functions. This theorem is also extended to the algebraic varieties V_k whose universal covering manifold is analytically equivalent to a bounded domain D of C^k on which the second problem of Cousin is solvable. *F. Gherardelli* (Florence)

3416:

Akizuki, Yasuo; and Matsumura, Hideyuki. On the dimension of algebraic system of curves with nodes on a non-singular surface. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 30 (1957), 143–150.

The problem treated here comes from the famous Anhang F of Severi's *Vorlesungen über algebraische Geometrie* [Leipzig, 1921] and is important for the discussion of the completeness of the characteristic series and of moduli of curves. Theorem: If E is the linear system on a nonsingular surface in a projective space that is cut out by all hypersurfaces of a given degree, then the algebraic subset of E consisting of all curves with $\geq d$ nodes and no other singularities, together with all specializations of these curves, is either empty or has each component of dimension $\geq \dim E-d$. The proof writes down the conditions that a hypersurface must satisfy to give an intersection of the desired sort, and shows that the imposition of one more node decreases the dimension of each component by at most one. *M. Rosenlicht* (Berkeley, Calif.)

3417:

Nakai, Yoshikazu. A property of an ample linear system on a non-singular variety. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 30 (1957), 151–156.

Same theorem as in the preceding review. The basis for the present proof is the consideration of the map of the given surface into a Grassmann variety gotten by first mapping into another projective space (via a given linear system) and then taking the tangent linear variety.

M. Rosenlicht (Berkeley, Calif.)

3418:

Matsumura, Hideyuki; and Nagata, Masayoshi. On the algebraic theory of sheets of an algebraic variety. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 30 (1957), 157–164.

Same theorem as in the preceding two reviews. The proof given here uses local algebra and Severi's original method of analytic branches. The main lemma states that the points of an r-dimensional variety V which decompose into $\geq d$ points on the derived normal model of V, together with all specializations of such points, form an algebraic set which, if V is of codimension one on a nonsingular variety, has each of its components of dimension $\geq r+1-d$. *M. Rosenlicht* (Berkeley, Calif.)

3419:

Zobel, A. On the contacts between the varieties of two systems. Rend. Mat. e Appl. (5) 17 (1958), 415–422.

Two algebraic varieties V^i $(i=1, 2)$ of dimensions v_i in r-dimensional projective space S_r are defined to have k-dimensional contact at a given point P, if P is a simple point on each of them, and their tangent spaces at P have in common a space S_k. If V^i belongs to a system T^i of dimension t_i, then, if

$$t_1+t_2+(v_1+v_2-r)(k+1)-k^2 = 0,$$

there is, under suitable conditions, a finite number of points at which some member of T^1 has k-dimensional contact with some member of T^2.

The author finds a formula for this number, μ, as the sum of products of complementary pairs of incidence numbers of the Schubert type. These numbers he obtains by using the representation, due to E. Martinelli [Atti Accad. Italia. Mem. **12** (1942), 917–943; MR **8**, 224], of space elements E_k ($E_k=S_k$ together with a point of itself) by points of a subvariety of $G\times S_r$, where G is the Grassmannian of S_k in S_r.

In the special cases in which V^1 is either a curve or a primal the author gives more explicit forms for the value of μ. In particular, in the classical case when $v_1=v_2=1$, $t_1=1$, $t_2=0$, $k=1$, $r=2$, i.e., when T^1 is a simply-infinite family of plane curves and T^2 is a single curve, the value reduces to the well-known form

$$\mu = \alpha m+\beta n$$

where α, β are the indices of the system T^1, and n, m are the order and class of T^2. *T. G. Room* (Sydney)

3420:

★Ribenboim, Paulo. O teorema de Riemann-Roch para curvas algébricas. [The Riemann-Roch theorem for algebraic curves.] Thesis, Faculdade Nacional de Filosofia. Rio de Janeiro, 1959. 82 pp.

Expository paper, clearly and well written, which follows the proof given in Chevalley's book [*Introduction to the theory of algebraic functions of one variable*, Amer. Math. Soc., New York, 1951; MR **13**, 64] with a few variants. *J. Dieudonné* (Paris)

3421:

Igusa, Jun-ichi. On the transformation theory of elliptic functions. Amer. J. Math. **81** (1959), 436–452.

A complete algebraic account of Kronecker's transformation theory of elliptic functions. Except for the exclusion of characteristic two all things done are compatible with specialization (including reduction mod p). Any elliptic curve is birationally equivalent to a Jacobi quartic $Y^2=1-2\rho X^2+X^4$, where $\rho\neq\pm1$, ∞, the natural map from these quartics to the space of isomorphism classes of elliptic curves being a galois covering with group S_3. Oversimplifying considerably, consider a rational map $(\rho, x, y)\to(\rho', x', y')$ from one such quartic to another, given by

$$x' = x^mF(x^{-1})F(x)^{-1}, \quad y' = G(x)F(x)^{-2}y,$$

where m is an odd integer and $F(X)$, $G(X)$ are polynomials, the former having constant term 1 and the expression for x' being in lowest terms. It is shown that the coefficients of $F(X)$, $G(X)$ are in $Q(\rho, \rho')$, Q being the prime field, and are integral over $Z[8\rho]$ if $\rho'=\rho$. Certain two-variable polynomials over Z that are canonically derived from the division points of given odd order (the quartic being considered an elliptic curve) are shown to be absolutely irreducible. Finally there is Kronecker's congruence relation: If the correspondence $(\rho, x, y)\to(\rho', x', y')$ is of odd prime degree p ($\neq$ characteristic of Q), then

$$F(X) = 1+\sum_{0<2i<p-1}\mu\gamma_iX^{p-2i-1}+\mu X^{p-1},$$

where $\gamma_1, \gamma_2, \cdots$ are integers of $Q(\rho, \mu)$ with reference to $Z[8\rho]$, and $N_{Q(\rho,\mu)/Q(\rho)}(\mu)=(-4/p)p$. *M. Rosenlicht* (Berkeley, Calif.)

3422:

Igusa, Jun-ichi. Fibre systems of Jacobian varieties. III. Fibre systems of elliptic curves. Amer. J. Math. **81** (1959), 453–476.

[For parts I and II, see same J. **78** (1956), 171–199, 745–760; MR **18**, 935, 936.] This paper gives an algebraic theory of elliptic modular functions and includes generalizations of a number of classical results due, among others, to Klein and Kronecker. First of all, a discussion of canonical forms for elliptic curves shows that it is possible to assign to each such curve A an element $j(A)$ of the universal domain such that $j(A)=j(B)$ if and only if A and B are isomorphic, and such that if an elliptic curve A' is a specialization of A (possibly unequal characteristics) then $j(A)$ has a unique specialization and this is $j(A')$. These properties show that $j(A)$ is contained in any field of definition of A and is uniquely determined up to the transformation $j\to\pm j+$integer; furthermore a model of A exists that is defined over $F(j(A))$, F being the prime field. In general, A has only two automorphisms, $a\to\pm a$, and the identification map, denoted by Ku if A is defined over K, maps A onto a projective line. Letting k be the algebraic closure of F, j a variable over k, A an elliptic curve defined over $k(j)$ and with $j(A)=j$, n an integer prime to the characteristic of k, Ω the n^2 points of A of order n, then the field $k(j, Ku(\Omega))$ is called the field of modular functions of level n. The main results state that $k(j, Ku(\Omega))$ is a galois extension of $k(j)$ with group the unimodular 2×2 integral matrices modulo n modulo its center; give the genus of $k(j, Ku(\Omega))$; and give a complete account of the ramification of this field over $k(j)$. *M. Rosenlicht* (Berkeley, Calif.)

3423:

Segre, Beniamino. Recenti prospettive nella teoria delle corrispondenze. Bull. Soc. Math. France **86** (1958), 355–372.

An expository lecture, dealing with various recent results in the theory of correspondences of various kinds. *J. A. Todd* (Cambridge, England)

3424:

Mizuno, Hirobumi. Sur les correspondances algébriques. Proc. Japan Acad. **34** (1958), 657–660.

Let V and V' be two complete non-singular varieties, A and A' be respectively the Albanese varieties of V and V', and α and α' be respectively the canonical mappings of V and V' into A and A'. Let $\dim V=n$, G be the additive group of $V\times V'$-cycles of dimension n, and G_a be the subgroup of G, consisting of X such that $S(\alpha'(X(x)-X(y)))=0$. The main theorem of this paper seems to be that the mapping $X\in G\to$linear extension of X (i.e., the homomorphism γ: $A\to A'$ such that $S(\alpha'(X(x)))=\gamma\cdot\alpha(x)+$constant) gives a monomorphism of G/G_a to the module of homomorphisms $H(A, A')$.

Some other theorems are proved, which are also immediate consequences of the existing theory of Picard and Albanese varieties. *T. Matsusaka* (Evanston, Ill.)

3425:

Lang, Serge. Reciprocity and correspondences. Amer. J. Math. **80** (1958), 431–440.

Let V, W be two complete varieties, non-singular in codimension 1. Let D be a divisor on the product $V\times W$ and $\mathfrak{a}$ and $\mathfrak{b}$ be respectively 0-cycles on V and W. If $D(\mathfrak{a})$ is defined and is the divisor of a function f on V, and if no component of $\mathfrak{b}$ is contained in a component of $D(\mathfrak{a})=(f)$, $f(\mathfrak{b})=\prod f(b_i)^{n_i}$ ($\mathfrak{b}=\sum n_ib_i$) is defined. If $\mathfrak{b}$ is of degree 0, $f(\mathfrak{b})$ is independent of the choice of f. Assume that $\mathfrak{a}$ and $\mathfrak{b}$ are both of degree 0 and that they are in the kernels of the canonical mappings of the groups of 0-cycles of degree 0 on V and W into the corresponding Albanese varieties. The main theorem of this paper asserts that if no component of $\mathfrak{a}\times\mathfrak{b}$ lies in the support of D, $D(\mathfrak{a},\mathfrak{b})$ and ${}^tD(\mathfrak{b},\mathfrak{a})$ are defined and equal. This is first proved on Abelian varieties and the general case is deduced from there. This is a generalization of Weil's reciprocity theorem $f((g))=g((f))$, where f and g are functions on a complete non-singular curve C, which can be obtained from this theorem by putting $V=W=C$, $D=$diagonal. Two applications of the main theorem are mentioned. (i) Let K be a finite Galois extension of a field k with the Galois group G, and A, B be Abelian varieties defined over k. Then a well-defined bilinear mapping of $H^1(G,A_K)$ and B_k into $H^2(G,K^*)$ is defined in terms of the operation $D(\ ,\ )$ (A_K [resp. B_k] is the group of rational points of A [resp. B] over K [resp. k], and K^* is the multiplicative group of K), generalizing the pairing of Tate on Jacobian varieties. (ii) Let A, B be Abelian varieties, D a divisor on the product $A\times B$, and $\mathfrak{a}$, $\mathfrak{b}$ two 0-cycles on A and B respectively, of degree 0, such that $nS(\mathfrak{a})=0$, $nS(\mathfrak{b})=0$. If no component of $\mathfrak{a}\times\mathfrak{b}$ lies in the support of D, ${}^tD(n\mathfrak{b},\mathfrak{a})/D(n\mathfrak{a},\mathfrak{b})=\varepsilon_{n,D}(a,b)$ is an nth root of unity, where $a=S(\mathfrak{a})$, $b=S(\mathfrak{b})$. $\varepsilon_{n,D}(a,b)$ depends only on the correspondence class of D and of a, b. On the other hand, $(n\delta)^{-1}({}^tD(\mathfrak{b}))$ is the divisor of a function f. When u is a generic point of A, $f(u+a)/f(u)=e_{n,D}(a,b)$ is also an nth root of unity. It is proved that $\varepsilon_{n,D}(a,b)=e_{n,D}(a,b)$, giving a complement to Kummer theory arising from the theory of divisorial correspondences. *T. Matsusaka* (Evanston, Ill.)

3426:

Snapper, Ernst. Cohomology theory and algebraic correspondences. Mem. Amer. Math. Soc. no. 33, 96 pp. (1959).

Let X_1, X_2 be two irreducible projective varieties defined over a field k, and X_3 an irreducible correspondence between X_1 and X_2 (i.e. an irreducible subvariety of $X_1\times X_2$). Using finite affine coverings of X_1 and X_2, one obtains a finite affine covering of $X_1\times X_2$, hence of X_3; this gives a double complex K such that, if X_1, X_2 are normal and if their function fields are algebraically in that of X_3, then the total cohomology groups of K are $H^n(X_3,L_3)$ and the cohomology groups of the edge-complexes of K are $H^n(X_i,L_i)$ ($i=1,2$), where L_i denotes the sheaf of local rings of X_i ($i=1,2,3$). A spectral sequence argument shows that, if X_3 is the graph of a regular rational mapping f of X_2 onto X_1, then $f^*:H^1(X_1,L_1)\to H^1(X_2,L_2)$ is a monomorphism (inequality of the sheaf-theoretic irregularities). The algebraic coherent sheaves over X_3 are then connected with the doubly graded finitely generated modules over the doubly graded coordinate ring of X_3; special attention is given to the sheaves arising from divisors on X_3, and applications to the theory of linear systems are given; the author uses the language of projective classes of modules [Snapper, Compositio Math. **13** (1956), 1–15, 16–38, 39–46; MR **18**, 671, 512, 513]. By using suitably chosen finite affine coverings of X_1 and X_2, one proves (with the above notations and hypotheses) that, if f is birational, then $h^r(X_1,L_1)\geq h^r(X_2,L_2)$, where r is the common dimension of X_1 and X_2. *P. Samuel* (Urbana, Ill.)

3427:

Nagata, Masayoshi. An example to a problem of Abhyankar. Amer. J. Math. **81** (1959), 501–502.

In a former paper [Ann. of Math. (2) **65** (1957), 268–281; MR **19**, 63], Shreeram Abhyankar suggested this problem: Let K be a function field of dimension 2 over an imperfect ground field k of characteristic p ($\neq 0$) and let k' be a purely inseparable extension of k of degree p. Let K' be the field generated by K over k'. Let R be a normal spot of K over k and let R' be the derived normal ring of R in K'. Assume that R' is a regular local ring. Is then R regular?

The author gives an example, satisfying the above conditions and two additional conditions, in which R is not regular. *T. R. Hollcroft* (Aurora, N.Y.)

3428:

Abhyankar, Shreeram. Tame coverings and fundamental groups of algebraic varieties. I. Branch loci with normal crossings; Applications: Theorems of Zariski and Picard. Amer. J. Math. **81** (1959), 46–94.

Let V be a normal projective variety (defined over an algebraically closed field k), K its function field, W an irreducible subvariety of V, K^* a finite separable algebraic extension of K, K' the least Galois extension of K containing K^*, V^* and V' the normalization of V in K^* and K', W^* an irreducible subvariety of V^* corresponding to W, and W' an irreducible subvariety of V' corresponding to W^*. Denote by R, R^*, R' the local rings of W in V, W^* in V^*, W' in V', and by M, M^*, M' their maximal ideals. The inertia group $G_i(W'/W)$ is the group of all automorphisms s of K'/K such that $s(x)-x\in M'$ for every x in R'. We say that W is unramified [resp. tamely ramified] in K' (or K^*) if the order of $G_i(W'/W)$ is 1 [resp. is not a multiple of the characteristic p of K; this is no restriction if p is 0]. Note that these definitions are distinct from those given in Abhyankar, same J. **79** (1957), 825–856 [MR **20** #872], but coincide with them if V is a curve. The union of all subvarieties W of V that are ramified (i.e., not unramified) in K^* is a closed subset $\Delta(K^*/V)$, called the branch-locus of V in K^*; this branch locus is distinct from V, and has local codimension 1 at every simple point of V. Various lemmas about non-ramification and tame ramification are given (e.g., transitivity properties, replacement of W by a subvariety, case of codimension 1). If $\Delta(K^*/V)$ is empty, we say that K^* is unramified over V and that V^* is an unramified covering of V; then every real valuation of K is unramified in K^*. If every subvariety (or point) of V is tamely ramified in K^*, we say that K^* is a tamely ramified extension of V.

Now let again V be a normal projective variety, K its function field, and W a closed subset of V (possibly empty). We denote by $\Omega_g(V-W)$ the family of all finite Galois extensions L of K such that $\Delta(L/V)\subset W$; by $\Omega_g'(V-W)$ [resp. $\Omega_g^*(V-W)$] the subfamily formed by those L that are tamely ramified over V [resp. such that $L:K$ is not a multiple of p]. The inverse system of the Galois groups of the members of $\Omega_g(V-W)$ [resp. $\Omega_g'(V-W)$, $\Omega_g^*(V-W)$] is called the fundamental [resp. tame fundamental, reduced fundamental] group tower of $V-W$, and is denoted by $\pi(V-W)$ [resp. $\pi'(V-W)$, $\pi^*(V-W)$]. More generally a group tower is an inverse system (G_i) of finite groups such that, for every i, the subsystem $(G_j)_{j\leq i}$ is isomorphic to the inverse system of all the factor groups of G_i. Several lemmas about group towers are given. A subgroup G' of the inverse limit G of a group tower (G_i) is said to be a weak parent group of (G_i) if it is dense in G (topologized by taking all the G_i's discrete); a weak parent group G' is called a parent group of (G_i) if the topology of G' induced by that of G coincides with the topology in which the subgroups of finite index form a basis for the neighbourhoods of the identity.

An important problem is to determine whether the tame fundamental group tower $\pi'(V-W)$ admits a finitely generated parent or weak parent group; this is not true for $\pi(V-W)$, the number of "untame" coverings being too large (here $p\neq 0$). In characteristic 0 all coverings are tame; by a theorem of Grauert and Remmert [C. R. Acad. Sci. Paris **245** (1957), 819–822, 882–885, 918–921; MR **19**, 1076, 1077] every finite topological covering V'' of $V-W$ may be completed to a normal algebraic variety V', which is a (ramified) covering of V; thus the topological fundamental group $\pi_1(V-W)$ is a (supposedly finitely generated) parent group of $\pi'(V-W)$.

In the abstract case, after giving some purely group-theoretic conditions for the existence of finitely generated weak parent groups and parent groups of a given tower, the author proceeds to prove the following results.

(1) Let V be a non-singular projective variety, and W a pure subvariety of codimension 1 of V, with t components W_j. We assume that V is simply connected (i.e., has no unramified covering), that each W_j moves in a linear system, and that W has only normal crossings (i.e., every point P of W is simple on all the analytic sheets of W at P, and the tangent hyperplanes of these sheets are linearly independent). Then $\pi'(V-W)$ has a weak parent group generated by t elements, and every weak parent group of $\pi'(V-W)$ is t-step nilpotent; furthermore, if the W_j's are pairwise connected, then $\pi'(V-W)$ is abelian. Taking for V a projective space (that is proved, by induction on the dimension, to be simply connected in the algebraic sense), one sees that the fundamental group of $V-W$ is abelian with t generators and one relation (all explicitly given); this generalizes to the abstract case a theorem of Zariski [Amer. J. Math., **51** (1929), 305–328].

(2) Add to the hypotheses of (1) the assumption that W is irreducible. Then $\Omega_g'(V-W)$ is finite, and its compositum K^* is a cyclic extension of K whose degree is projectively determined by V and W. Denoting by V^* the normalization of V in K^* and by f its natural mapping onto V, $V^*-f^{-1}(W)$ is tamely simply connected; the normalization of V in any field between K and K^* is simply connected. As a consequence one gets an abstract version of a theorem of Picard which says that any cyclic surface in projective 3-space is simply connected.

{The paper contains many misprints, all of them of a trivial nature.} *P. Samuel* (Urbana, Ill.)

3429:

Nakano, Shigeo. Tangential vector bundle and Todd canonical systems of an algebraic variety. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29** (1955), 145–149.

In this paper the author proves in detail the following result, which was announced as a supplementary result at the end of a preceding paper [J. Math. Soc. Japan **7** (1955), 1–12; MR **17**, 409]: For a non-singular algebraic variety in a projective space, the Chern classes of the tangential vector bundle coincide with the Todd canonical systems. For notation, including the definition of the universal bundle $\mathfrak{R}$ over the Grassman manifold H, one may refer to the preceding paper mentioned above. Then the proof may be described briefly as follows. Letting V^r denote a non-singular algebraic variety of dimension r in a complex projective space L^N, one obtains a natural mapping Φ of V^r into $H(r+1, N+1)$, and if $\Omega_{(p)}$, $p=1,\cdots,r+1$, denote the Chern classes of $\mathfrak{R}$, then the characteristic polynomial of $\Phi^{-1}(\mathfrak{R})$ is $F(\lambda)=\lambda^{r+1}+c_1'\lambda^r+\cdots+c_{r+1}'$, where $c_p'=\Phi^{-1}(\Omega_{(p)})$. If X denotes (the cohomology class attached to) a general hyperplane section of V, then $F(\lambda-X)=\lambda^{r+1}+t_1\lambda^r+\cdots+t_{r+1}$, where (by definition) $t_1,\cdots,t_r$ ($t_{r+1}=0$) are the Todd canonical systems (cycles) of V. If $\mathfrak{B}$ is the complex line bundle attached to the divisor X, one finds by direct computation with transition functions that $\Phi^{-1}(\mathfrak{R})=\mathfrak{B}\otimes\mathfrak{N}+\mathfrak{F}$, where $\mathfrak{N}$ is a (topologically) trivial complex line bundle and $\mathfrak{F}$ is the tangential bundle over V. Using this fact, and using $c_1,\cdots,c_p$ to denote the Chern classes of $\mathfrak{F}$, one obtains immediately, by comparing the coefficients of characteristic polynomials, that $c_i=t_i$, $i=1,\cdots,p$. *W. L. Baily, Jr.* (Munich)

LINEAR ALGEBRA

See also 3774, 3937.

3430:

Englefield, M. J. Oriented flat submanifolds in an affine space. Canad. J. Math. **10** (1958), 535–546.

Let V_n denote an n-dimensional vector space; let $V_m\subset V_n$. Two ordered bases $a_1,\cdots,a_m$ and $b_1,\cdots,b_m$ of V_m are said to assign the same [the opposite] inner orientation to V_m if $b_i=\sum_1^m\beta_i^k a_k$ and $|\beta_i^k|>0$ [$|\beta_i^k|<0$]. If V_n' is the dual space of V_n, let $V'_{n-m}\subset V_n'$ consist of those linear forms which vanish in V_m. An "outer orientation" of V_m is an inner orientation of V'_{n-m}.

Now let E_n be an affine n-space. If E_m is a linear subspace of E_n, any inner [outer] orientation of the vector space parallel to E_m defines one in E_m. Let E_p and E_q be two linear subspaces of E_n; $E_s=E_p\cap E_q$; E_t = space spanned by E_p and E_q. Suppose E_p, E_q and E_s [and E_t] are given inner [outer] orientations. Using tensor algebra the author discusses the elementary problem of assigning an inner [outer] orientation to E_t [to E_s].

P. Scherk (Toronto, Ont.)

3431:

Demaria, Davide Carlo. Sulle equazioni generali di primo grado in un corpo sghembo. Boll. Un. Mat. Ital. (3) **14** (1959), 36–41.

Theorem: If the division ring C is n-dimensional over its center Γ then the transformations $x \to \sum_1^n a_i x b_i + c$, where $a_i, b_i, c \in C$, coincide with the affine transformations of the Γ-vector space C. *M. Rosenlicht* (Berkeley, Calif.)

3432:

Marcus, Marvin; and Newman, Morris. On the minimum of the permanent of a doubly stochastic matrix. Duke Math. J. **26** (1959), 61–72.

Let K_n be the convex hull of the set of $n \times n$ permutation matrices and let $K_n{}^0$ be the interior of K_n. For any $n \times n$ matrix X let

$$\operatorname{per}(X) =_{\mathrm{df}} \sum_\sigma \prod_{i=1}^n X_{i\sigma(i)},$$

where the summation runs over the permutations σ of $(1, 2, 3, \cdots, n)$. Theorem: If $\operatorname{per}(A) = \min_{S \in K_n} \operatorname{per}(S)$ and $A \in K_n{}^0$, then $\operatorname{per}(A) = n!/n^n$ and each entry in A is $1/n$. This, modulo the hypothesis $A \in K_n{}^0$, answers a question raised by van der Waerden [Jber. Deutsche Math. Verein. **35** (1926), Abt. 2, 117]. *S. Sherman* (Philadelphia, Pa.)

3433:

Dulmage, A. L.; and Mendelsohn, N. S. The term and stochastic ranks of a matrix. Canad. J. Math. **11** (1959), 269–279.

Suppose $A = (a_{ij})$ is an $n \times n$ matrix all of whose entries are non-negative. Let $S = \sum_{1 \leq i \leq n,\ 1 \leq j \leq n} a_{ij}$. Let

$$M = \max\left\{ \sum_{1 \leq j \leq n} a_{ij},\ \sum_{1 \leq j \leq n} a_{ji} : 1 \leq i \leq n \right\}.$$

Let the term rank, $\rho(A)$, be the order of the largest minor of A with a non-zero term in the expansion of its determinant. Theorem: $\rho(A)$ is not less than the least integer greater than or equal to S/M. "B is an (r, T) d.s. extension of A" means that B is an $(n+r) \times (n+r)$ matrix formed by adding r rows and r columns to A, all entries in B are non-negative, and all row and column sums of B are equal to T. Theorem: If $r \geqq n$, there is an (r, T) d.s. extension of A. If $r \leqq n-1$, then $M \leqq T \leqq S/(n-r)$ is a necessary and sufficient condition that there be an (r, T) d.s. extension of A. The (r, T) d.s. extensions of A form a convex set whose vertices are given graphical interpretation. The stochastic rank of A is the maximum σ such that, for some T, A has a $(n-\sigma, T)$ d.s. extension. $\sigma = [S/M]$. Connections between ρ and σ are derived. *S. Sherman* (Philadelphia, Pa.)

3434:

Jack, Henry; and Macbeath, A. M. The volume of a certain set of matrices. Proc. Cambridge Philos. Soc. **55** (1959), 213–223.

In the space of all real $n \times n$ matrices we consider the subset M of all matrices τ with determinant 1 and with norm $\|\tau\| \leqq k$ (k is a given real number, and the norm is defined as the supremum of the length of τx, where x runs through all vectors of length 1). M^* denotes the cone $\bigcup_{0<\lambda\leqq 1} \lambda M$. The authors determine the volume of M^*. First they show, using the representation $\tau = u_1 \delta u_2$ (u_1, u_2 orthogonal, δ diagonal), that

$$\mu(M^*) = 2^{-\frac{1}{2}n(n-1)}[\nu(U)]^2 J,$$

where $\nu(U)$ is the volume of the orthogonal group, J is the integral

$$J = \int_R \prod_{i>j} (\delta_i{}^2 - \delta_j{}^2)\, d\delta_1 \cdots d\delta_n,$$

and R is described by

$$0 < \delta_1 \cdots \delta_n \leqq 1, \quad \delta_1 \leqq k(\delta_1 \cdots \delta_n)^{1/n}.$$

In the next step, J is expressed in terms of a single integral along a complex contour, which can be evaluated by residue calculus.

{The reviewer remarks that the integral J is of the type studied in his paper in J. Indian Math. Soc. **19** (1955), 133–151; MR **18**, 121; especially section 9.}

N. G. de Bruijn (Amsterdam)

3435:

Ostrowski, Alexander. Über Eigenwerte von Produkten Hermitescher Matrizen. Abh. Math. Sem. Univ. Hamburg **23** (1959), 60–68.

It is well known that if A, B are hermitian $n \times n$ matrices and A is positive definite or positive semi-definite, then all characteristic roots of AB are real. The main object of the present paper is to obtain more detailed quantitative information about these characteristic roots. We denote by m, M the least and greatest characteristic roots of A, by $\lambda_1 \leqq \cdots \leqq \lambda_n$ the characteristic roots of B, and by $\Lambda_1 \leqq \cdots \leqq \Lambda_n$ those of AB. It is then proved that there exist numbers Θ_ν such that

$$\Lambda_\nu = \Theta_\nu \lambda_\nu, \quad m \leqq \Theta_\nu \leqq M \quad (1 \leqq \nu \leqq n).$$

The principal tool in the proof is the following generalization of the Fischer-Courant minimax principle. Let B, P be hermitian $n \times n$ matrices and suppose that P is positive definite. If $\rho_1 \leqq \cdots \leqq \rho_n$ denote the (necessarily real) roots of the equation $\det(xP - B) = 0$, then

$$\rho_\nu = \max_L \min_\xi \xi^* B\xi / \xi^* P\xi \quad (1 \leqq \nu \leqq n),$$

where the minimum is taken with respect to all vectors ξ in a given vector subspace L and the maximum is taken with respect to all subspaces L of fixed dimension $n - \nu + 1$. A direct proof of this result had been given earlier by F. R. Gantmacher [*Teoriya matric*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953; MR **16**, 438; pp. 259–261]; here it is exhibited as a simple consequence of the Fischer-Courant principle.

Next, the author obtains several inequalities concerning the moduli of the characteristic roots of AB, where A and B are now assumed to be normal. In particular, he gives a new derivation of a theorem on unitary matrices due to R. Brauer and A. Loewy [Tôhoku Math. J. **32** (1930), 44–49].

The paper concludes with the proof of the following results. Let H, H_1 be hermitian $n \times n$ matrices. Then

$$(*) \qquad \xi^* H\xi > |\xi^* H_1 \xi| \quad \text{for all } \xi \neq 0$$

if and only if H is positive definite and all characteristic roots of $H^{-1}H_1$ are less than 1 in modulus. Furthermore, if the characteristic roots of H and H_1 are denoted by

$$\alpha_1 \leqq \cdots \leqq \alpha_n, \quad \beta_1 \leqq \cdots \leqq \beta_n$$

respectively and if (*) is valid, then

$$\alpha_\nu > \max(\beta_\nu, -\beta_{n-\nu+1}) \quad (1 \leqq \nu \leqq n).$$

For the special case when H is real and symmetric while

iH_1 is real and skew-symmetric, these results reduce to theorems obtained by O. Taussky-Todd [J. Washington Acad. Sci. **47** (1957), 263–264; MR **19**, 725].

L. Mirsky (Sheffield)

3436:

Nudel'man, A. A.; and Švarcman, P. A. The spectrum of the product of unitary matrices. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 111–117. (Russian)

Let A, B be unitary matrices. In general there are no restrictions on the possible roots of AB. However, if the roots of A lie on an arc shorter than p, the roots of B lie on an arc shorter than q, and $p+q<2\pi$, the following theorem is valid. Let $0\leqq\varphi_1\leqq\cdots\leqq\varphi_n<2\pi$ be the arguments of the roots of A; $0\leqq\psi_1\leqq\cdots\leqq\psi_n<2\pi$ be the arguments of the roots of B; $\varphi_n-\varphi_1+\psi_n-\psi_1<2\pi$. Then if $0\leqq\omega_1\leqq\cdots\leqq\omega_n<2\pi$ are the roots of AB, the point $(\omega_1, \cdots, \omega_n)$ is contained in the intersection of the two convex sets spanned respectively by the points $(\varphi_1+\psi_a, \varphi_2+\psi_b, \cdots, \varphi_n+\psi_z)$ and $(\psi_1+\varphi_a, \psi_2+\varphi_b, \cdots, \psi_n+\varphi_z)$, where $(a, b, \cdots, z)$ runs through all possible permutations of $(1, 2, \cdots, n)$.

J. L. Brenner (Palo Alto, Calif.)

3437:

Cremer, H.; und Effertz, F. H. Über die algebraischen Kriterien für die Stabilität von Regelungssystemen. Math. Ann. **137** (1959), 328–350.

The stability of the solutions of linear systems of differential equations with constant coefficients reduces to the algebraic problem of determining the location of the roots of the characteristic equation. Since the original work of Routh and Hurwitz a great deal of effort has been devoted to this problem. This paper is a survey of known results, together with a discussion of old and new interconnections between different techniques and a presentation of some new results. Anyone interested in either the algebraic aspects, the analytic aspects, or the applications to control theory, will want to read this paper.

R. Bellman (Santa Monica, Calif.)

3438:

Mott, J. L.; and Schneider, Hans. Matrix algebras and groups relatively bounded in norm. Arch. Math. **10** (1959), 1–6.

Abstract norms $\nu(A)$ of finite matrices A with complex elements have been studied extensively in recent years [see e.g. Faddeeva, *Vyčislitel'nye metody lineĭnoĭ algebry*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950; transl. by C. D. Benster, Dover, New York, 1959; MR **13**, 872; **20** #6777]. It can be shown that always $\nu(A)\geqq\rho(A)=\max|\lambda_A|$ where λ_A is a characteristic root of A. Sets of matrices which are "relatively bounded" are studied here, i.e., ν/ρ is bounded on the set for some ν. A norm which coincides with ρ on the set is called "minimal" on the set. Matrices A with a norm minimal on A alone are characterized in several ways [see also Householder, Oak Ridge Nat. Lab. Rep. **ORNL-1883** (1955); MR **17** 790]. Two equivalent necessary and sufficient conditions are found to ensure that a whole algebra of $n\times n$ matrices possesses a norm which is minimal on it. They are (1) that the algebra is relatively bounded, (2) that all elements can be transformed simultaneously to diagonal form by a similarity and hence that the algebra is commutative. The proof for this is based on a theorem due to Motzkin and Taussky [Trans. Amer. Math. Soc. **80** (1955), 387–401; MR **19**, 242; see also Drazin, Canad. J. Math. **8** (1956), 341–354; MR **17**, 1179]. The existence of a minimal norm is also studied for groups of matrices, not necessarily diagonable or commutative.

O. Taussky-Todd (Pasadena, Calif.)

3439:

Egerváry, E. Über eine konstruktive Methode zur Reduktion einer Matrix auf die Jordansche Normalform. Acta Math. Acad. Sci. Hungar. **10** (1959), 31–54. (Russian summary, unbound insert)

The main object of this paper is to find a brief constructive method for reducing a matrix to Jordan normal form. The method is practicable for computing. Previously a constructive method was given by Farahat [J. London Math. Soc. **32** (1957) 178–180; MR **19**, 242], but this method is shorter because of the symmetry of the procedure, which uses left and right eigen and principal vectors [cf. e. g. R. Zurmühl, *Matrizen*, Springer, Berlin-Göttingen-Heidelberg, 1950; 2nd ed., 1958; MR **12**, 73; **20** #4567]. Two numerical examples, one derogatory, one not derogatory, are worked through.

O. Taussky-Todd (Pasadena, Calif.)

3440:

Robinson, D. W. A note on a simple matrix isomorphism. Math. Mag. **32** (1958/59), 213–215.

From the isomorphism between the field of the complex numbers $\alpha=a+bi$ and the field of the matrices

$$A=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

follows the correspondence $f(\alpha)\leftrightarrow f(A)$ for a scalar function $f(z)$ if and only if $f(\bar\alpha)=\overline{f(\alpha)}$. To generalize this result consider the ring of all complex $n\times n$-matrices $u=(\alpha_{\mu\nu})$ and the isomorphic ring of the corresponding $2n\times 2n$-matrices $U=\varphi(u)=(A_{\mu\nu})$. It is shown that for a scalar function $f(z)$ one has $\varphi(f(u))=f(\varphi(u))$ if and only if $f^{(k)}(\bar\alpha_j)=\overline{f^{(k)}(\alpha_j)}$ $(k=0, 1, \cdots, s_j-1)$ where α_j $(j=1, \cdots, r)$ are the characteristic roots of u and s_j their multiplicities in the minimum polynomial of u respectively.

H. Schwerdtfeger (Montreal, P.Q.)

3441:

Wild, Jonathan. On the discriminants of a bilinear form. Canad. Math. Bull. **2** (1959), 46–48.

Continuing his paper in the same Bull. **1** (1958), 180 [MR **21** #677], the author considers non-symmetric bilinear forms $f(x, y)$ defined on a vector space E of dimension n over a field of characteristic different from two. He defines rank f as $n-d$, where d is the dimension of the subspace of those vectors for which $f(x, y)=0$ for all $y\in E$; and index f as the maximum dimension of a subspace in which f vanishes identically. Now the author calls the derived symmetric bilinear form

$$g(x, y)=f(x_0, x_0)\cdot f(x, y)-f(x_0, x)\cdot f(x_0, y)$$

the discriminant of f at the point x_0 $(f(x_0, x_0)\neq 0)$. He proves that rank $g=$rank $f-1$, and index $g=$index $f+1$ [$=$index f] if there exists [does not exist] a subspace of maximal dimension in which f vanishes identically, contained in the subspace of all x such that $f(x, x_0)=0$.

J. Aczél (Debrecen)

3442:

Levine, Jack. Note on an identity of Cayley. Amer. Math. Monthly **66** (1959), 290–292.

The following theorem is proved: Let $|a_{ij}|$ be a determinant of order $n \leqq 4$ and let $\overset{+}{|}a_{ij}\overset{+}{|}$ denote the corresponding permanent. If the rank of the matrix $[a_{ij}]$ is less than 3, and if all $a_{ij} \neq 0$, then

$$\overset{+}{\left|}\frac{1}{a_{ij}}\overset{+}{\right|}\left|\frac{1}{a_{ij}}\right| = \left|\frac{1}{a_{ij}^2}\right|.$$

The case $n=3$ was first proved by Cayley. The method of proof for $n=4$ is fairly computational and makes use of a formula of Muir which gives an expression for the product of a general permanent and determinant.

J. K. Goldhaber (St. Louis, Mo.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 3352, 3431, 3438.

3443:

Schenkman, Eugene. Some remarks on the multiplicative group of a sfield. Proc. Amer. Math. Soc. **9** (1958), 231–235.

Let K be a division ring. Let S be a division subring, S' the group of units, $Z(S)$ the centralizer, and $N(S)$ the normalizer of S in K. Theorem: If (1) S is subinvariant in K, that is, if S' is a member of a composition series of K'; and if $S \subseteq Z(S)$; then (2) either $S=K$, or else $S \subseteq Z(K)$. The author attributes the method of proof to R. Brauer. {The following information was communicated by the author in a letter to the reviewer. Although it is stated in the paper that (1) implies (2) for non-commutative S satisfying card $S \cap Z(K) > 5$, W. R. Scott has pointed out to the author that the case $N(N(S))=N(S)$ is not considered. This gap will be bridged, and the condition on $S \cap Z(K)$ will be removed, in a note to appear, with W. R. Scott as co-author with the present author.}

C. C. Faith (Heidelberg)

3444:

De Cicco, John. Some theorems concerning commutative rings with unit which admit involutorial automorphisms. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **92** (1957/58), 225–242. (Italian summary)

Let R be a commutative ring with identity in which $2\neq 0$. If R has an involutorial automorphism, let X be the set of all self-conjugate elements of R and Y the set of all skew-conjugate elements; then X is a subring of R and Y is an additive subgroup of R. Various relations between X and Y are discussed. Now if X is a commutative ring with identity in which $2\neq 0$, then the set Q of elements of the form $z=x+iy$, where x and y are in X and i is in Q but not in X and satisfies the equation $i^2=\alpha+i\beta$, is a commutative ring with identity. Q is called a quadratic extension of X and Q has an involutorial automorphism given by conjugation, if $z=x+iy$, $\bar{z}=x+\beta y-iy$. Conditions under which a given commutative ring R with involutorial automorphism is a quadratic extension of the ring X of self-conjugate elements are given and, in particular, characterizations of quadratic extensions of a field are obtained. These results are applied to a discussion of the associated z-plane, and polynomial forms and functions over a commutative ring with involutorial automorphism are discussed.

E. H. Batho (Cambridge, Mass.)

3445:

★Aubert, K. E. Une théorie générale des idéaux et ses applications. Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18–23 août 1957, pp. 28–35. Mercators Tryckeri, Helsinki, 1958. 209 pp. (1 plate)

The author observes that there is a formal similarity among certain aspects of the theory of ideals of commutative rings, various notions of ideal in the theory of lattices and ordered sets, the notion of a band in a reticulated space, and the theory of perfect differential ideals of commutative differential rings. His purpose is to find a single formalism which encompasses all these cases. He considers a commutative demi-group (monoid) D written multiplicatively and supposes given a mapping $A \to A_x$ of the set of subsets of D into itself such that: (1) $A \subset A_x$; (2) $A \subset B_x \Rightarrow A_x \subset B_x$; (3) $A \cdot B_x \subset B_x \cap (A \cdot B)_x$. (1) and (2) imply that $A_{xx}=A_x$; A_x is called the x-ideal generated by A. After describing various conditions equivalent to the second half of (3), he points out that if D is of finite character, i.e., A_x is always equal to the union of the x-ideals generated by the finite subsets of A, then Krull's theorem that the radical of an ideal is the intersection of its prime divisors generalizes. If it is further assumed that every irreducible ideal is primary then the theory of ideals in noetherian rings also generalizes. Similarly for the theory of prime decomposition in Dedekind rings. Other generalizations are considered (space of maximal ideals, algebra of continuous functions on a compact space, valuations of ordered groups). A detailed exposition is promised for a subsequent paper.

E. R. Kolchin (New York, N.Y.)

3446:

Ishikawa, Takeshi. On Dedekind rings. J. Math. Soc. Japan **11** (1959), 83–84.

A characterization of Dedekind domains by functors Ext and Hom, that an integral domain Λ is a Dedekind ring if and only if $\mathrm{Hom}_\Lambda(A, C)$ is divisible whenever A is torsion-free and C is divisible, is shown in this paper.

The assertion itself, but not the proof, is analogous to that given by Hattori [same J. **9** (1957), 381–385; MR **20** #8546] for Prüfer rings.

M. Nagata (Kyoto)

3447:

Mori, Yoshiro. On the integral closure of an integral domain. VI. Bull. Kyoto Gakugei Univ. Ser. B **13** (1958), 1–3.

[For parts IV and V, see same Bull. **10** (1957), 1–5; **11** (1957), 1–7; MR **19**, 938.] Let R be an integral domain. An F-ideal A in R is an ideal such that $A^{-1}A=A$ and $(A^{-1})^{-1}=A$; then $R \subset A^{-1} \subset R'$ (R': integral closure of R). Any non-zero F-ideal of R is the conductor in R of some overring of R. If R has no other F-ideals than itself and (0), then it is integrally closed. Every maximal F-ideal is prime. For the conductor of R' in R to be $\neq (0)$, it is necessary and sufficient that the F-ideals in R satisfy the descending chain condition.

P. Samuel (Urbana, Ill.)

3448:

Berger, Robert. Zur Idealtheorie analytisch normaler Stellenringe. J. Reine Angew. Math. **201** (1959), 172–177.

Let A be a noetherian local domain of dimension d. If A is analytically irreducible, every power of its maximal ideal contains a prime ideal of height $d-1$. In case A is a power series ring $R[[X_1, \cdots, X_{d-1}]]$ over a discrete valuation ring R, one takes prime ideals of the form $(X_1^{n(1)}-u^n, \cdots, X_{d-1}^{n(d-1)}-u^n)$ (u uniformizing element of R; $n, n(1), \cdots, n(d-1)$ pairwise relatively prime large integers). One passes then to the case in which A is complete, using the fact that A is a finite module over a ring of the preceding kind. The general case follows, provided suitable precautions have been taken in the preceding steps. *P. Samuel* (Urbana, Ill.)

3449:

Augier, Jacques. Filtration d'un anneau local déduite d'une valuation. C. R. Acad. Sci. Paris **248** (1959), 3514–3516.

Let A be a local integral domain of rank (=Krull dimension) d. Let $\mathfrak{m}$ and K be the maximal ideal and the field of quotients of A respectively. Assume that B is a discrete valuation ring, with maximal ideal $\mathfrak{p}$, of K, dominating A. Set $\mathfrak{m}_n=\mathfrak{p}^n\cap A$, and let $\varphi(n)$ be the largest integer such that $\mathfrak{m}^{\varphi(n)}$ contains $\mathfrak{m}_n$. With these notations, the author proves the following easy assertions.

Theorem 1. If $d\geqq 2$ and if $B/\mathfrak{p}$ is algebraic over $A/\mathfrak{m}$, then there is an integer N such that $\varphi(n)=\varphi(N)$ for any $n\geqq N$.

Theorem 2. Assume that A is the formal power series ring $k[[X_1, \cdots, X_n]]$ in indeterminates X_i over a field k, and that the valuation v defined by B is homogeneous, namely, the value $v(f)$ ($f\in A$) is the minimum of values of terms of f. Then $\varphi(n)$ is the smallest integer not less than $n/(\max v(X_i))$.

{Reviewer's remark: One can prove very easily the following assertion, which is a generalization of theorem 1: If $\varphi(n)$ is not bounded, then the transcendence degree of $B/\mathfrak{p}$ over $A/\mathfrak{m}$ is exactly $d-1$.} *M. Nagata* (Kyoto)

3450:

★Samuel, Pierre. Progrès récents de l'algèbre locale. Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956, pp. 231–243. Centre Belge de Recherches Mathématiques. Établissements Ceuterick, Louvain; Librairie Gauthier-Villars, Paris; 1957. 293 pp. 250 francs belges.

The author reports in this article what happened in the theory of local rings during 1950–55.

The topics are: (i) the lemma of Artin-Rees; (ii) the introduction of the homological method; and (iii) other results (not stated in detail) by the reviewer, Northcott and Rees, and Geddes and Narita. *M. Nagata* (Kyoto)

3451:

Lenz, Hanfried. Lineare Halbgruppen mit beschränkten Eigenwerten. Math. Ann. **137** (1959), 150–166.

Let (V, G) be an n-dimensional vector space over the field of complex numbers which admits as linear operators the elements of a semigroup G. G is called irreducible if (V, G) (as an additive group with operators) is simple. (V, G) is unitary if G leaves invariant a nondegenerate positive definite Hermitian form; (V, G) is multiply unitary if every factor space of a composition series is a simple unitary space.

The author studies semigroups with bounded eigenvalues; i.e., all eigenvalues of all elements of G have absolute value $\leqq 1$. In particular, if all absolute values of the eigenvalues are 1 (e.g., when G is a group with bounded eigenvalues), then (V, G) is unitary if G is completely reducible, and multiply unitary if G is reducible. More general results are given, and similar results are obtained when the field of scalars is taken to be the real numbers.

For the linear mapping $\alpha\in G$, define the norm $\|\alpha\|=\max |a_{ij}|$, where (a_{ij}) is the matrix of α relative to some basis. G is bounded if the set of the norms of its elements is bounded. The author shows that if all the cyclic subgroups of a group G of linear transformations on real n-space V are bounded, then so is G, provided that either (1) G is completely reducible, or (2) all composition factors of (V, G) have at most dimension 2.

B. N. Moyls (Vancouver, B.C.)

3452:

Kertész, A. A remark on the general theory of modules. Publ. Math. Debrecen **6** (1959), 86–89.

An extension R_1 of an associative ring R is said to have property P if it has a unit element and if for any R-module G it is possible to extend the operation of R on G to R_1 in such a way that G becomes a unitary R_1-module. R_1 is a minimal extension with property P if no ring between R and R_1 has this property. The author characterizes the usual adjunction of an identity to R, the set of all ordered pairs (r, n) where $r\in R$ and n is an integer, as the only minimal extension of R with property P.

D. K. Harrison (Philadelphia, Pa.)

3453:

Kertész, Andor. Investigations in the theory of operator modules. I, II. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **8** (1958), 411–436, **9** (1959), 15–50. (Hungarian)

The main results of these papers are published in Acta Math. Acad. Sci. Hungar. **8** (1957), 235–257; Acta Sci. Math. Szeged **18** (1957), 207–234; **19** (1958), 251–252 [MR **21** #1321; **19**, 1155; **20** #6441] and the paper reviewed above. *L. Fuchs* (Budapest)

3454:

Fort, Jacques. Quelques propriétés des sous-modules tertiaires d'un module sur un anneau non nécessairement commutatif. C. R. Acad. Sci. Paris **248** (1959), 1748–1750.

The author extends to submodules of a module the theory of decomposition of left ideals in a ring due to Lesieur and Croisot [same C. R. **243** (1956), 1988–1991; MR **18**, 637]. If U is a left module with minimum or maximum condition over a ring with left minimum condition, then every submodule of U is an intersection of tertiary submodules with the usual uniqueness. A submodule (still over a ring with minimum condition) is tertiary if and only if it has a unique left residual which is prime. The definitions are the analogues of Lesieur and Croisot's. No proofs are given. *D. Zelinsky* (Evanston, Ill.)

3455:

Fried, E. On Galois modules of vector spaces. Publ. Math. Debrecen **6** (1959), 101–110.

Let Γ be a finite-dimensional algebra over a field K, let L be a right Γ-module and M the ring of all K-endomorphisms of L. If $d(JM)/d(M)=d(J)/d(\Gamma)$ for all right ideals J in Γ, where d denotes dimension over K, the author calls Γ "a Galois module of L". {Reviewer's comment: It can be shown that this condition is equivalent to: a direct sum of copies of the dual $\mathrm{Hom}_K(L, K)$ of L is a free Γ-module; the condition is satisfied by $L=\Gamma$ if and only if Γ is a Frobenius algebra.} In this case, there is perfect duality between the right ideals of Γ and their annihilators in L.

Using Wedderburn theory, the author proves that if Γ is a Galois module both of L and of Γ and if $d(\Gamma)=d(L)$ then Γ is isomorphic to L as a Γ-module. If L is a finite-dimensional normal separable field extension of K and Γ is the group algebra over K of the Galois group, then Γ is easily seen to be a Galois module of both Γ and of L. Hence $\Gamma \cong L$, which is the normal basis theorem.

D. Zelinsky (Evanston, Ill.)

3456:

Gemignani, Giuseppe. Osservazioni relative alla dipendenza algebrica su un anello. Ricerche Mat. **7** (1958), 235–240.

The author considers a number of theorems concerning algebraic and transcendental extensions of rings and shows, by means of counter-examples, that certain hypotheses cannot be omitted from the statements of these theorems. For example, in the theorem "If θ is algebraic over the ring A and A is semi-simple, every element of $A[\theta]$ is algebraic over A" the assumption that A is semi-simple cannot be omitted. Another example is the following theorem: For any ring A there exists a super-ring B algebraic over A and an element θ algebraic over B such that θ is transcendental over A.

H. W. Brinkmann (Swarthmore, Pa.)

3457:

Steinfeld, Ottó. Ein Beweis des Wedderburn-Artinschen Strukturensatzes. Magyar Tud. Akad. Mat. Kutató Int. Közl. **3** (1958), no. 1/2, 63–65. (Hungarian and Russian summaries)

This is a not self-contained exposition of the Wedderburn-Artin structure theorem for "Halbeinfache Ringe". The author's claim to "ein kurzer elementarer Beweis" is weakened by the assumption at the outset of a Hilfssatz (accredited to Emmy Noether by the author) which is no small missing link. The author makes no claims for the originality of the presentation. {No reference is made to Emmy Noether's original paper [Math. Z. **30** (1929), 641–692], where the Hilfssatz and other substantial similarities may be found. (Cf. the author's elements δ_i, δ_i^*, with E. Noether's elements c_{i1}, c_{1i} [loc. cit., p. 666], respectively.)}

C. C. Faith (Heidelberg)

3458:

Babič, A. M. The Levitzki radical. Dokl. Akad. Nauk SSSR **126** (1959), 242–243. (Russian)

The sum of all the locally nilpotent ideals of a ring K is a locally nilpotent ideal, and is called the Levitzki radical of K. The following theorem is proved: In any ring K, the Levitzki radical L of K is the intersection of all the ideals P of K such that the factor ring K/P is a primitive L-semisimple ring.

R. Ree (New York, N.Y.)

3459:

Hsieh, Pang-chieh. On S-irreducible algebras. Sci. Record (N.S.) **2** (1958), 323–325.

Let $\mathfrak{A}$ be an associative algebra with a unity element and finite dimension over a field $\mathfrak{F}$, and S be an automorphism over $\mathfrak{F}$ of $\mathfrak{A}$. Then $\mathfrak{A}$ is said to be S-irreducible if $\mathfrak{A}$ is not the direct sum $\mathfrak{B}+\mathfrak{C}$ of ideals $\mathfrak{B}=\mathfrak{B}S$ and $\mathfrak{C}=\mathfrak{C}S$. $\mathfrak{A}$ is known to be S-irreducible if and only if

$$\mathfrak{A} = \mathfrak{B}+\mathfrak{B}S+\mathfrak{B}S^2+\cdots+\mathfrak{B}S^{t-1}$$

for ideals $\mathfrak{B}S^t$ such that $\mathfrak{B}=\mathfrak{B}S^t$ is irreducible. Hence $\mathfrak{A}$ is S-irreducible if and only if $\mathfrak{A}$ is the direct product $\mathfrak{A}=\mathfrak{Y}\times\mathfrak{D}$, where $\mathfrak{D}$ is the diagonal algebra of dimension t and is unique, and $\mathfrak{Y}$ is isomorphic to $\mathfrak{B}$; $\mathfrak{Y}=\mathfrak{Y}S^t$ has an automorphism T induced by S^t. If also $\mathfrak{A}=\mathfrak{Y}_0\times\mathfrak{D}$ with $\mathfrak{Y}_0=\mathfrak{Y}_0S^t=\mathfrak{Y}_0T_0$ then the algebra-group pair $(\mathfrak{Y}, T)$ and $(\mathfrak{Y}_0, T_0)$ are isomorphic. In *Structure of Algebras* [Amer. Math. Soc. Colloq. Publ., New York, 1939; MR **1**, 99; p. 80, l. −9] the reviewer stated incorrectly that S^t induces an automorphism T_0 in $\mathfrak{Y}_0$ in all cases, and based the statement on the fact that there is always an idempotent $e=eS^t$ such that the mapping induced by $ye \rightarrow y_0e$ goes into $(yS^t)e \rightarrow (y_0S^t)e$. This implies that the automorphism T of $\mathfrak{Y}$ is mapped onto the isomorphism T_0 of $\mathfrak{Y}_0$ onto $\mathfrak{Y}_0S^t$. The author produces a simple counter-example to show that it is possible that $\mathfrak{Y}_0S^t \neq \mathfrak{Y}_0$; he shows that S^t induces an automorphism in any $\mathfrak{Y}_0$ if and only if every automorphism over $\mathfrak{F}$ of $\mathfrak{Y}$ commutes with T, and that the application to cyclic semi-fields, for which this whole theory was produced, is then correct.

A. A. Albert (Chicago, Ill.)

3460:

Kupisch, Herbert. Beiträge zur Theorie nichthalbeinfacher Ringe mit Minimalbedingung. J. Reine Angew. Math. **201** (1959), 100–112.

The paper studies the structure of a ring R (containing 1) with minimum condition, firstly under each of the following conditions which together are characteristic for a quasi-Frobenius ring: (A) the left and the right annihilators $l(N)$, $r(N)$ of the radical N are equal; (B) for every primitive idempotent e the left [resp. right] ideal Re [resp. eR] has only one minimal subideal. Under (A), it is shown, every irreducible component in the mth factor N^{m-1}/N^m of the upper Loewy series of the R-left-module R has an isomorphic image in the mth factor $r(N^m)/r(N^{m-1})$ of the lower Loewy series. Under the stronger condition that (C) $l(\mathfrak{a})=r(\mathfrak{a})$ for every two-sided ideal $\mathfrak{a}$, Loewy series of Re, eR are studied. Loewy series of Re, eR are considered also under the condition (B), to yield a refinement of a result of Thrall [Trans. Amer. Math. Soc. **64** (1948), 173–183; MR **10**, 98] on Cartan invariants. The paper studies also generalized uni-serial indecomposable rings and shows, for example, that in case they are algebras over an algebraically closed field they are determined, up to isomorphisms, by some simple invariants.

T. Nakayama (Nagoya)

3461:

Hebroni, P. Ueber die inneren Automorphismen des abstrakten Differentialringes. Compositio Math. **14**, 77–82 (1959).

Let D be a differential ring in the sense of the author's earlier paper [Compositio Math. **5** (1938), 403–429]; this is essentially a ring in which operations of differentiation and

integration are defined in a formal algebraic manner. An element $\alpha \in D$ is said to be perfect if α^{-1}, α' and $(\alpha^{-1})'$ are in D. An element $\alpha \in D$ is called a transforming element if it is perfect and $\alpha'\alpha^{-1}$ commutes with all differentiable elements of D. For variable $x \in D$ and a fixed transforming element $\alpha \in D$, the mapping $x \to \alpha x \alpha^{-1}$ is an automorphism of D in the sense of preserving all relevant structures. Such a mapping is called an inner automorphism of the differential ring. The inner automorphisms form a multiplicative group. *W. E. Jenner* (Lewisburg, Pa.)

3462:

Feller, Edmund H. Primary intersections for two sided ideals of a Noetherian matrix ring. Trans. Amer. Math. Soc. **90** (1959), 336–339.

Let D be a Noetherian ring and D_n be the ring of all $n \times n$ matrices over D. Let $I = I_1 \cap I_2 \cap \cdots \cap I_s$ be an irredundant representation of an ideal I in D as an intersection of irreducible ideals I_a of D. The author shows that $I_n = \bigcap_{k=1}^{n} \bigcap_{a=1}^{s} (I_a, k)$ is a representation of $I_n (\subseteq D_n)$ as an irredundant intersection of irreducible right ideals (I_a, k) in D_n. The right ideal (I_a, k) is the set of all matrices whose kth row contains only elements of I_a, the rest being arbitrary. He extends this result to representations of primary right ideals and to sets of representations, following the notions of his earlier paper [Trans. Amer. Math. Soc. **81** (1956), 342–357; MR **17** 1047].

S. A. Amitsur (New Haven, Conn.)

3463:

Batho, Edward H. Some remarks on non-commutative extensions of local rings. Nagoya Math. J. **14** (1959), 45–51.

Let S be a commutative, Noetherian, semilocal ring with radical J and let R be an S-algebra. If S is complete, if $\bigcap J^n R = 0$, and if R/JR is a finitely generated (S/J)-module then R is a finitely generated S-module and a complete semilocal ring. If S is local but not necessarily complete, if Q is an ideal in R with $Q \cap S$ J-primary, then R/Q satisfies the minimum condition.

D. Zelinsky (Evanston, Ill.)

3464:

Jans, J. P. On Frobenius algebras. Ann. of Math. (2) **69** (1959), 392–407.

An algebra A with 1 over a field K is defined to be a Frobenius algebra if A is self-dual relative to a non-degenerate inner product, with values in K, satisfying $(xy, z) = (x, yz)$ for all $x, y, z \in A$ and moreover A has an automorphism α satisfying $(x, y) = (y, x^\alpha)$ for all $x, y \in A$. The inner product induces two topologies on A, and an automorphism as above exists if and only if these two topologies are equivalent. If an algebra A is Frobenius with respect to two inner products (x, y), $\langle x, y \rangle$ with associated automorphisms α, β, and if the topologies induced by the two inner products are equivalent, then there exists a regular element $c \in A$ such that $\langle x, y \rangle = (x, yc)$ and $x^\beta = cx^\alpha c^{-1}$ for all $x, y \in A$. In a Frobenius algebra A the "annihilator relations" $l(r(L)) = L$, $r(l(R)) = R$ for left [right] closed ideals L [R] hold. Also, A is "self-injective" in a certain topological sense. The intersection F of all closed maximal right ideals of finite index in A coincides with the intersection of all closed maximal (two-sided) ideals of finite index, and the "top" A/F of A is dual to (and looks like) the "bottom" of A. If the ideal B generated by all minimal idempotent ideals is finite-dimensional, A is the direct sum $B + B^\perp$ where $B^\perp$ is bound, in Hall's sense. The paper studies further when a factor algebra of a Frobenius algebra is Frobenius. Also Ikeda's characterization of Frobenius algebras by Shoda's condition is extended to the present infinite-dimensional case.

T. Nakayama (Nagoya)

3465:

Bergmann, Artur. Über den Zusammenhang von Norm und Spur und ihrer Funktionalgleichungen in gewissen Algebren und dessen Anwendung auf Ringe mit Spurenbedingungen. Math. Nachr. **19** (1958), 237–254.

This is a generalization of the author's earlier paper [Math. Nachr. **15** (1956), 55–76; MR **18**, 109]. Let A be an associative algebra with unit element over a field K, not necessarily of finite dimension. In case the dimension is finite, it is assumed throughout that the characteristic of K is either zero or greater than the dimension of A over K. Suppose first that A has finite dimension over K and that the trace and norm functions, S and N, are defined as usual in terms of the regular representation. These functions have the following properties:

$$(\mathrm{S}_1) \qquad S(x+y) = Sx + Sy,$$

$$(\mathrm{S}_2) \qquad n! \det \mathfrak{S}_n(xy) = \det \mathfrak{S}_n(x) \cdot \det \mathfrak{S}_n(y),$$

$$(\mathrm{S}_3) \qquad S(\xi \cdot x) = \xi Sx \quad (\xi \in K, x \in A),$$

$$(\mathrm{N}_1) \qquad N(xy) = Nx \cdot Ny,$$

$$(\mathrm{N}_2) \qquad (n+1)!(x_1, \cdots, x_{n+1}) = 0,$$

$$(\mathrm{N}_3) \qquad (\xi x, \cdots, y, \cdots, z)_n = (x, \cdots, \xi y, \cdots, z)_n = (x, \cdots, y, \cdots, \xi z)_n = \xi(x, \cdots, y, \cdots, z)_n \quad (\xi \in K),$$

$$(*) \qquad Nx = (n!)^{-1} \det \mathfrak{S}_n(x),$$

$$(**) \qquad Sx = ((n-1)!)^{-1}(\Delta^{n-1} Nx - \Delta^{n-1} N(0)) \equiv n(x, 1, \cdots, 1)_n.$$

The bracket symbol is defined inductively by $(x) = Nx$ and

$$(k+1)!(x_1, \cdots, x_k, x_{k+1}) = k!(x_1, \cdots, x_{k-1}, x_k + x_{k+1}) - k!(x_1, \cdots, x_{k-1}, x_k) - k!(x_1, \cdots, x_{k-1}, x_{k+1}) \quad \text{for} \quad k \geq 1.$$

$\mathfrak{S}_n(x)$ denotes the matrix

$$\begin{bmatrix} Sx & 1 & 0 & & 0 \\ Sx^2 & Sx & 2 & 0 & 0 \\ \vdots & \vdots & Sx & 3 & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ Sx^{n-1} & & & & n-1 \\ Sx^n & Sx^{n-1} & \cdots & \cdots & Sx \end{bmatrix};$$

and

$$\Delta^k f(x) = \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} f(x+\nu).$$

Theorem: Let A be an algebra with unit element over K. Suppose a function S is given satisfying the conditions (S_1), (S_2), (S_3) and $S1 = n$. Then the function N defined by $(*)$ satisfies (N_1), (N_2) and (N_3). Conversely suppose a function N is given satisfying (N_1), (N_2), (N_3) and the further condition that (N_2) holds for no natural number smaller than $n+1$. Then the function S defined by $Sx = n(x, 1, \cdots, 1)_n$ satisfies (S_1), (S_2), (S_3) and $S1 = n$.

Theorem: Let A be an algebra with unit element over a field K and having a trace function satisfying (S_1), (S_2), also $S1=n$ (n=natural number) and $(x, y_1, \cdots, y_{n-1})\not\equiv 0$ for $x\neq 0$ in A and the y_k arbitrary in A. Then every element of A satisfies a polynomial equation of degree n over K. This theorem is applied to algebras with non-degenerate trace functions of the type arising in the theory of complex multiplication. In particular, it is possible to prove the finite-dimensionality of the multiplication algebra of a function field of one variable. This depends on obtaining an independent proof of (S_2); and the author adds in a footnote that this has been accomplished. {It would be interesting to know if a proof of (S_2) can be given for the multiplication algebra of an arbitrary (abstract) abelian variety. If so, it seems that the author's techniques (in this and his previous paper, loc. cit.) will yield a proof that the multiplication ring of an abelian variety is a finitely-generated module over the integers. It is reasonably clear that such a proof would not be simpler than those currently available. On the other hand, it could very possibly shed new light on some of the problems of complex multiplication.}

W. E. Jenner (Lewisburg, Pa.)

3466:

Jongmans, F. Étude algébrique des matrices de Hodge. Mém. Soc. Roy. Sci. Liège (5) **1** (1958), no. 2, 7–68.

Exposition of a large part of Albert's work on Riemann matrices, slightly generalized, with emphasis on clarity. A long introductory section states most of the necessary preliminaries from the theory of algebras.

M. Rosenlicht (Berkeley, Calif.)

3467:

Berman, Gerald; and Silverman, Robert J. Simplicity of near-rings of transformations. Proc. Amer. Math. Soc. **10** (1959), 456–459.

The authors prove that the near-rings consisting of all the transformations of a group into itself and of all those transformations of a group into itself which leave the group identity fixed, with addition and multiplication defined by the group operation and by iteration, respectively, are simple in that they have no nontrivial homomorphic images. Also all the invariant sub-near-rings of these near-rings are determined. (The sub-near-ring P of the near-ring Q is said to be invariant in Q if $PQ\subseteq P$ and $QP\subseteq P$.) *W. E. Deskins* (East Lansing, Mich.)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

3468:

Ree, Rimhak. Note on generalized Witt algebras. Canad. J. Math. **11** (1959), 345–352.

The author continues his investigation of the Lie algebras of prime characteristic p called generalized Witt algebras [Trans. Amer. Math. Soc. **83** (1956), 510–546; MR **18**, 491], to determine their derivations. Theorem: Every derivation is the sum of an inner derivation and a derivation whose matrix relative to a fixed standard basis is diagonal, provided the base field is infinite and $p>2$. {The conclusion seems to carry over to finite fields, by using the author's result and an argument about extension of the base field.} In the second part of the paper the author presents a generalization of these algebras which includes most of the simple Lie algebras previously considered by Albert and Frank [Univ. e Politec. Torino. Rend. Sem. Mat. **14** (1955), 117–139; MR **18**, 52] and by Jennings and Ree [Trans. Amer. Math. Soc. **84** (1957), 192–207; MR **18**, 583].

G. B. Seligman (New Haven, Conn.)

3469:

Kleinfeld, Erwin. Quasi-nil rings. Proc. Amer. Math. Soc. **10** (1959), 477–479.

The nucleus N of a ring R is the subring of all $n\in R$ such that $(n, R, R)=(R, n, R)=(R, R, n)=0$, where (x, y, z) denotes the associator $(xy)z-x(yz)$. The author shows that if R is a nonassociative prime ring, of characteristic different from 2, then R is anticommutative (i.e., $r^2=0$ for all $r\in R$) if $r^2\in N$ for all $r\in R$. If in addition $(x, y, z)+(y, z, x)+(z, x, y)$ is in N for all $x, y, z\in R$, then R is a Lie ring. *H. Minc* (Vancouver, B.C.)

3470:

Minc, H. Theorems on nonassociative number theory. Amer. Math. Monthly **66** (1959), 486–488.

The free logarithmetic $\mathfrak{L}$ arose in the reviewer's papers [Proc. Roy. Soc. Edinburgh Sect. A **59** (1939), 153–162; **62** (1949), 442–453; **64** (1955), 150–160; MR **10**, 677; **17**, 825] as the arithmetic of indices of powers in a general nonassociative algebra, and has been studied on number-theoretic (and other) lines by the author [ibid. **64** (1957), 319–341; MR **19**, 836] and by T. Evans [Amer. Math. Monthly **64** (1957), 299–309; MR **20** #58]. The following three theorems are proved. The author, quoting a referee, observes that (1) and (2) do not involve addition and actually apply to free multiplicative semigroups. (Capital letters denote indices of $\mathfrak{L}$.) (1) $PQ=QP$ if and only if P and Q are powers of the same index. (2) If $X^mP=Y^n$ or $PX^m=Y^n$ where P is prime and m, n are integers greater than 1, then X and Y are both powers of P. (3) If $X^p+Y^q=Z^r$ where p, q, r are integers greater than 1, then $X=2^k$, $Y=2^m$, $Z=2^n$ and $kp=mq=nr-1$. ("2" is here an index.) From (3) follows the analogue of Fermat's last theorem, also obtained by Evans: $X^r+Y^r=Z^r$ implies $r=1$. *I. M. H. Etherington* (Edinburgh)

HOMOLOGICAL ALGEBRA

See also 3446, 3481, 3836.

3471:

Nakayama, Tadasi. On algebras with complete homology. Abh. Math. Sem. Univ. Hamburg **22** (1958), 300–307.

One says that an algebra A over a field K has complete homology if there exists an exact sequence

$$(1)\qquad 0\to A\to X_{-1}\to X_{-2}\to\cdots$$

of $A-A$-homomorphisms with X_p $A-A$-projective (i.e., X_p is a two-sided A-module or a left $A\otimes A^*$ module, where A^* is the dual algebra of A, and is $A\otimes A^*$-projective).

The author has previously shown [Poincaré Colloquium, Paris, 1954; Nagoya Math. J. **13** (1958), 115–121; MR **20** #6450] that if A is a quasi-Frobenius algebra, then it has complete homology. (A quasi-Frobenius algebra is one which has the property that the right A-module $A^0 = \mathrm{Hom}_K(A, K)$ is a projective right A-module.)

The paper is devoted to proving some special theorems which strengthen the author's conviction that an algebra which has complete homology is necessarily quasi-Frobenius. Two conditions are considered along with the existence of an exact sequence (1):

I. An exact sequence (1) should exist with X_p not only $A-A$-projective, but also $A-A$-injective;

II. Every finite exact sequence of $A-A$-homomorphisms

$$(2) \qquad 0 \to A \to X_{-1} \to X_{-2} \to \cdots \to X_{-n}$$

with X_p all $A-A$-projective can be extended to a similar exact sequence $0 \to A \to X_{-1} \to \cdots \to X_{-n} \to X_{-n-1}$.

These stronger conditions are fulfilled by quasi-Frobenius algebras.

Let $1 = \sum_{\mu,i} e_i^{(\mu)}$ be a decomposition of the unit element 1 of A into mutually orthogonal primitive idempotents, where $Ae_i^{(\mu)} \simeq Ae_j^{(\nu)}$ when and only when $\mu = \nu$. A is said to have minimal right ideals of arbitrary type if, for every μ, A has a minimal right ideal isomorphic to $e_1^{(\mu)}A/e_1^{(\mu)}N$, where N denotes the radical of A. The author proves that if A has minimal right ideals of arbitrary type, and if there exists an exact sequence $0 \to A \to X$ with X $A-A$-projective, then A is a quasi-Frobenius algebra.

Other theorems indicating when conditions I and/or II imply that the algebra is quasi-Frobenius are also proved here. The author suggests that it may be worthwhile to classify algebras A according to the maximal number n such that an exact sequence (2) exists, or to the smallest number n for which an extension as described in II fails.

D. Buchsbaum (Providence, R.I.)

3472:

Nakayama, Tadasi. Higher dimensional cohomology groups in generalized quaternion algebras. Abh. Math. Sem. Univ. Hamburg **23** (1959), 174–179.

Let A be a division algebra with center a $\mathfrak{p}$-adic number field k and with $[A:k] = n^2$. Let W be a maximal subfield of A, of degree n and unramified over k. Let $\mathfrak{o}$ and I be the integers of k and W respectively and let $\mathfrak{O}$ be the maximal order of A. The first result is that for any prime ideal $\mathfrak{P}$ of $\mathfrak{O}$ and any integer r,

$$H^i(\mathfrak{O}, \mathfrak{O}/\mathfrak{P}^r) = H^i(\mathfrak{O}, I; \mathfrak{O}/\mathfrak{P}^r).$$

Here the first group is the usual Hochschild cohomology group for the $\mathfrak{o}$-algebra $\mathfrak{O}$, while the second one is the relative group introduced by Nakayama [Duke Math. J. **19** (1952), 51–63; MR **13**, 620] and Hochschild [Trans. Amer. Math. Soc. **82** (1956), 246–269; MR **18**, 278]. Next, let A be a generalized quaternion division algebra over k. Then $H^i(\mathfrak{O}, \mathfrak{O}/\mathfrak{P}^r) = H^1(\mathfrak{O}, \mathfrak{O}/\mathfrak{P}^r)$. Finally the author combines these results by the usual globalization techniques with the results of Kawada [Sci. Papers Coll. Gen. Ed., Univ. Tokyo **2** (1952), 1–8; MR **14**, 348] to obtain a description of $H^i(\mathfrak{O}, \mathfrak{O}/\mathfrak{A})$, where A is now a generalized quaternion algebra over an algebraic number field, $\mathfrak{O}$ is a maximal order of A, and $\mathfrak{A}$ is an ideal of $\mathfrak{O}$.

A. Rosenberg (Evanston, Ill.)

3473:

Golod, E. The cohomology ring of a finite p-group. Dokl. Akad. Nauk SSSR **125** (1959), 703–706. (Russian)

Let G be a finite p-group and R an associative and commutative ring on which G acts as a group of ring-homomorphisms. Then the cohomology group $H(G, R)$ can be turned into an associative ring by introducing in it a multiplicative structure, denoted by $\cup$, with the anti-commutativity rule: if ξ and η are homogeneous cohomology classes of dimension p and q, then $\xi \cup \eta = (-1)^{pq} \eta \cup \xi$. In the present note the author considers the cases when R is Z or Z_{p^n} and G acts on these rings trivially. He proves the following two theorems. (1) The cohomology rings $H(G, Z)$ and $H(G, Z_{p^n})$ are finitely generated. (2) Let K be a cyclic normal subgroup of the centre of G. Then the Hochschild-Serre spectral sequence [Trans. Amer. Math. Soc. **74** (1953), 110–134; MR **14**, 619] becomes stationary, i.e., $E_r = E_{r+1} = \cdots = E_\infty$ for sufficiently large r. The first theorem leads to an affirmative answer to a question raised in a paper by Kostrikin and Šafarevič [Dokl. Akad. Nauk SSSR **115** (1957), 1066–1069; MR **19**, 1156]: the Poincaré function $R_G(t)$ of the algebra $H(G, Z_{p^n})$ is a rational function.

K. A. Hirsch (London)

3474:

Rim, Dock Sang. Modules over finite groups. Ann. of Math. (2) **69** (1959), 700–712.

Let π be a finite group. The first main result says that the following properties for a π-module A are equivalent: (i) A is cohomologically trivial; (ii) for each prime p there exists an integer i such that $H^i(\pi', A) = H^{i+1}(\pi', A) = 0$ for a p-Sylow group π' of π; (iii) A has projective dimension $\leqq 1$ over the integral group ring $Z(\pi)$ of π; (iv) similarly to (iii) with $< \infty$ replacing $\leqq 1$; (iii′) or (iv′) similarly to (iii) or (iv), with "injective" replacing "projective". Further, if one of π-modules A, B is cohomologically trivial, then

$$H^n(\pi', A \otimes B) \cong H^{n+2}(\pi', \mathrm{Tor}_1^Z(A, B))$$

for all n and all subgroups π' of π; similarly with $\otimes, +,$ Tor replaced by Hom, $-$, Ext. Both these theorems are far-going generalizations and improvements of the reviewer's [Ann. of Math. (2) **65** (1957), 255–267; Nagoya Math. J. **12** (1957), 171–176; MR **19**, 841; **20** #4587] results. The proof starts with observations on weakly projective π-modules, including their characterization by the property $H(\pi, \mathrm{Hom}_Z(A, A)) = 0$, which is not only applied to the further part of the proof but is later combined with the above result to yield another interesting characterization of weakly π-projective modules. In the second half of the paper, the isomorphism classes of finitely generated projective modules over a ring Λ are considered modulo classes of free ones, to yield the projective class group $\Gamma(\Lambda)$. On the basis of Chevalley's [*L'arithmétique dans les algèbres de matrices*, Actualités Sci. et Industr. No. 323, Hermann, Paris, 1936] results on modules over a Dedekind ring and those of Diederichsen [Abh. Math. Sem. Hamburg **13** (1938), 357–412] and Reiner Proc. Amer. Math. Soc. **8** (1957), 142–146; MR **18**, 717] on integral representations of cyclic groups, $\Gamma(\Lambda)$ in the case of the integral group ring $\Lambda = Z(\pi)$ of a cyclic group π of prime order p is proved to be isomorphic to the ideal class group of the field of pth roots of 1; this settles a problem of Cartan and Eilenberg [*Homological algebra*, Princeton Univ. Press, Princeton, N.J., 1956; MR **17**, 1040].

T. Nakayama (Nagoya)

3475:

Hilton, P. J.; and Ledermann, W. Homology and ringoids. II. Proc. Cambridge Philos. Soc. **55** (1959), 149–164.

[Part I: same Proc. **54** (1958), 152–167; MR **20** #7050a.] The paper reformulates the basic concepts of category and abelian category in a language closer to group and ring theory. In consequence the language of principle ideals, annihilators, etc., can be advantageously introduced and used to give better insight into some of the standard arguments that have to be carried out in abelian categories. As an example, the authors proceed to develop the rudiments of homology theory through the exact homology sequence for complexes in an abelian category. The result justifies the impression that the search for a more advantageous method should not be abandoned.

S. Eilenberg (New York, N.Y.)

GROUPS AND GENERALIZATIONS

See also 3342, 3361, 3445, 3470, 3473, 3517, 3518.

3476:

Schenkman, Eugene. The equation $a^n b^n = c^n$ in a free group. Ann. of Math. (2) **70** (1959), 562–564.

The equation in the title is shown, for $n>1$, to imply that a, b and c lie in a cyclic subgroup. The problem reduces to the case that n is prime. Supposing that a and b do not commute implies the existence of a, b and c satisfying the same relation in $H=F/F^{p+1}$ for F free on inverse images of a and b, and F^{p+1} the $(p+1)$st term of the descending central series for F. A contradiction is obtained by constructing a homomorphism of H onto a group G, a split extension of an elementary n-group by a cyclic group of order n, which maps a^n and b^n into 1, but cannot map any c^n, for c congruent to ab modulo H', into 1. [Other proofs: M. P. Schützenberger, C. R. Acad. Sci. Paris **248** (1959), 2435–2436; MR **21** #2000; J. Stallings, Notices Amer. Math. Soc. **6** (1959), 532; G. Baumslag, J. London Math. Soc. **35** (1960), 30–32.]

R. C. Lyndon (Princeton, N.J.)

3477:

Rapaport, Elvira Strasser. Note on Nielsen transformations. Proc. Amer. Math. Soc. **10** (1959), 228–235.

Take two isomorphic groups G, H, presented as factor groups of isomorphic free groups. The isomorphisms of G, H which are induced by isomorphisms of the free groups are called Nielsen isomorphisms. Examples show that an isomorphism of G, H need not be a Nielsen isomorphism with respect to every presentation of these groups. However, every isomorphism is a Nielsen isomorphism with respect to some presentation (depending on the isomorphism) where the common rank of the free groups involved is twice the minimum number of generators of G or H.

The second part of the paper is concerned with groups whose commutator subgroups are free groups of rank two while their factor commutator groups are infinite cycles. Such a group can always be presented by two generators and a single relation, and in any such presentation it is a Nielsen isomorphic image of one of three (nonisomorphic) groups, each of them being considered in a particular presentation.—The fundamental group L of Listing's knot is one of these groups; also, in the first part of the paper an automorphism of L serves as an example of non-Nielsen isomorphisms.

{The paper lacks clarity in several places. Some arguments have gaps, but the reviewer thinks that these can probably be filled. The statement in lines 6 and 5 from below on page 223 is false but irrelevant. There are a number of obvious misprints.}

L. G. Kovács (Keele)

3478:

Wiegold, James. Nilpotent products of groups with amalgamations. Publ. Math. Debrecen **6** (1959), 131–168.

The author combines two ideas: the theory of generalized free and direct products of groups with amalgamations [B. H. Neumann, Philos. Trans. Roy. Soc. London Ser. A **246** (1954), 503–557; MR **16**, 10] and the theory of nilpotent products and verbal products of groups [Golovin, Mat. Sb. (N.S.) **27** (**69**) (1950), 427–454; MR **12**, 672; Moran, Proc. London Math. Soc. (3) **6** (1956), 581–596; MR **20** #3908]. Let gfp, gnp, gvp stand respectively for generalized free product(s) with amalgamations, generalized nilpotent product(s) with amalgamations, and generalized verbal product(s) with amalgamations. gnp and gvp are defined and examples given. As in the case of the gfp, the gnp (or gvp) may not exist. If a particular gnp (or gvp) exists, then the corresponding gfp exists, but not necessarily vice versa. Schreier [Abh. Math. Sem. Univ. Hamburg **5** (1927), 161–183] proved that if all the given groups intersect in a single group, then the corresponding gfp exists. The author shows that there is no corresponding theorem for gnp (or gvp), and constructs many counter-examples to dismiss conjectures and to indicate the difficulties of finding necessary and sufficient conditions for the existence of his products. Generalized free nilpotent [verbal] products are defined, and necessary and sufficient conditions are found for the existence of the generalized free GN_2 product of two groups A and B with H amalgamated. GN_2 products are the author's generalization of Golovin's metabelian products [Mat. Sb. (N.S.) **28** (**70**) (1951), 431–444; MR **13**, 105]. Necessary conditions are given for the existence of GN_3 products (corresponding to Golovin's second nilpotent products) of two groups with one subgroup amalgamated.

R. R. Struik (Vancouver, B.C.)

3479:

Marchionna Tibiletti, Cesarina. Rappresentazioni classiche e moderne dei gruppi astratti. Rend. Sem. Mat. Fis. Milano **28** (1959), 107–123. (English summary)

This is an expository lecture, starting from the definition of groups, and describing permutation representations and the "produit complet" (now usually called "wreath product") of Krasner and Kaloujnine [Acta Sci. Math. Szeged **13** (1950), 208–230; **14** (1951), 39–66, 69–82; MR **14**, 242]. The representation of abstract groups as subgroups of wreath products receives special attention.

B. H. Neumann (Manchester)

3480:

Szép, J. Über eine allgemeine Erweiterung von Gruppen. I. Publ. Math. Debrecen **6** (1959), 60–71.

The author poses and solves the following extension problem [generalizing the extension theory of Schreier,

Monatsh. Math Phys. **34** (1926), 165–180; and the skew products of L. Rédei, J. Reine Angew. Math. **188** (1950), 201–227; MR **14**, 13]: Let A be a group with unit element ϵ and Γ a set with a distinguished element ε; the problem is to determine all group structures on the product set $A\times\Gamma$ such that (1) the mapping $a\rightarrow a\times\varepsilon$ of A into $A\times\Gamma$ is a homomorphism and (2) the different right cosets of the image in $A\times\Gamma$ are just the sets $A\times\alpha$ $(\alpha\in\Gamma)$.—For simplicity write $a\alpha$ for $a\times\alpha$ $(a\in A,\ \alpha\in\Gamma)$, identify a with $a\varepsilon$ and α with $\epsilon\alpha$, and define formal products αa and $\alpha\beta$ (for any $a\in A$, α, $\beta\in\Gamma$) by writing $\alpha a={}^{\alpha}a\alpha^{a}$, $\alpha\beta={}^{\alpha}\beta\alpha^{\beta}$, where α^{a}, α^{β} are given functions with values in Γ and ${}^{\alpha}a$, ${}^{\alpha}\beta$ are given functions with values in A. Under the above identifications $\varepsilon=\epsilon$, which leads to certain normalization conditions on the given functions. Assuming these, the author gives the necessary and sufficient conditions on the given functions for $G=A\times\Gamma$ to be a group with respect to the multiplication $(a\alpha)(b\beta)=a((\alpha b)\beta)$. The set Γ is a set of two-sided coset representatives if and only if for each $\alpha\in\Gamma$ there is just one β satisfying $\alpha^{\beta}=\epsilon$. Further, the relation between this extension and the Schreier extension are described and the maximal normal subgroup of G contained in A is characterized as the kernel of the homomorphism $\pi: a\rightarrow\pi_a$ of A into the group of permutations of Γ, where $\pi_a=\binom{\alpha}{\alpha^a}$. *P. M. Cohn* (Manchester)

3481:

Harrison, D. K. Infinite abelian groups and homological methods. Ann. of Math. (2) **69** (1959), 366–391.

Homological algebra has reached the place where, in abelian group theory, it can be used to obtain numerous, new significant results of a purely group-theoretic nature. All groups mentioned are abelian. After setting the homological stage in § 1, where such notions which are subsequently needed are brought together, the author goes on, in § 2, to discuss co-torsion groups, those groups which have no non-trivial extensions by torsion-free groups. For the group Q of rationals, $\mathrm{Hom}(Q,G)=0$ and $\mathrm{Ext}(Q,G)=0$ characterize the co-torsion groups. Let Z be the group of integers. The correspondence $\mathrm{Hom}(Q/Z,\quad)$ is one-to-one on the torsion, divisible groups to the torsion-free, co-torsion groups, and the latter are precisely those groups which are isomorphic to a direct summand of an unrestricted direct sum (direct product) of p-adic integers. Adjusted co-torsion groups are those with no torsion-free direct summands, precisely those whose torsion-quotient groups (the groups divided out by their torsion subgroups) are divisible. Every co-torsion group is uniquely a direct sum of a torsion-free and of an adjusted co-torsion group. The correspondence from adjusted co-torsion groups to their torsion subgroups is one-to-one on all the former to all reduced torsion groups. If a group G is reduced, there is a monomorphism from G to $\mathrm{Ext}(Q/Z,G)$, a co-torsion group. If S is a torsion group and if X is a torsion-free group, a one-to-one correspondence is set up from the set of mixed groups M with torsion subgroup S and torsion-quotient group X to $\mathrm{Hom}(Q\otimes X,\ \mathrm{Ext}(Q/Z,S)/S)$.

By using exact sequences properly with the Pontrjagin duality theory, the author is able to characterize algebraically, in § 3, those abelian groups which are totally disconnected and compact in at least one topology, and similarly for the connected compact case. The principal result is that the abelian groups which are compact in at least one topology are precisely those groups which are direct products of finite cyclic groups, groups of p-adic integers, the reals and groups of type $Z(p^\infty)$, where the number of these last does not exceed the number of copies of the reals for any prime p. A topology is introduced into a group G by taking absolute values of elements: $|a|=10^{-m}$, where m is the largest (possibly infinite) integer for which $a\in m!G$. The author characterizes direct summands of direct products of finite cyclic groups as those groups G in which every Cauchy sequence has a limit with the further property that $\bigcap n!G=0$ (over all positive integers n). A torsion-free group is such a direct summand of such a direct product if and only if it is a direct summand of a direct product of copies of the p-adic integers. In § 4, he shows that if A is a torsion group and if B is any abelian group, then $\mathrm{Hom}(A,B)$ is a direct summand of a direct product of finite cyclic groups.

In a final section on torsion-free groups X, the dimension, $f(X)$, is defined in terms of the totality of vector space dimensions of the X/pX; rank, $n(X)$, as the dimension of $Q\otimes X$; and further, one considers the ranks of all the $\mathrm{Hom}(A_p,X)$, where A_p is the subgroup of those elements of Q which have the form a/p^j, and forms an associated rank, $e(X)$, from these. If $n(X)=f(X)e(X)<\infty$, the author calls X regular. If $pX=X$ for all but a finite number of primes p, then regularity is shown to be the same as being a direct sum of a finite number of rank one groups. The paper concludes with a result which shows that there is an "immense number of torsion-free groups ..., even for very small rank". *F. Haimo* (St. Louis, Mo.)

3482:

Fuchs, L. On character groups of discrete abelian groups. Acta Math. Acad. Sci. Hungar. **10** (1959), 133–140. (Russian summary, unbound insert)

The problem of determining the algebraic structure of compact abelian groups was studied by Kaplansky and Hulanicki, and the latter gave a complete description of abelian groups which can be made compact [Bull. Acad. Polon. Sci. Cl. III **4** (1956), 405–406; Fund. Math. **44** (1957), 156–158, 192–197; Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 71–73; MR **19**, 1063; **20** #2398]. The author gives here a new proof to Hulanicki's results, using Pontrjagin's duality theorem only to the effect that the class of character groups of discrete abelian groups coincides with the class of all compact abelian groups. He observes first that the group of reals mod 1 is, as an abstract group, the complete direct sum of quasi-cyclic groups $Z(p^\infty)$, one for each prime p, and that the character group of a discrete abelian group G is the complete direct sum of $\mathrm{Hom}\,(G,Z(p^\infty))$ with p running over all primes. The author then proves Hulanicki's results by studying the algebraic structure of the groups $\mathrm{Hom}(G,Z(p^\infty))$ in three steps, for torsion-free, for torsion and for mixed group G. The dependence of the structure of the character group of G on certain cardinal invariants of G is also studied. *K. Iwasawa* (Cambridge, Mass.)

3483:

Gruenberg, K. W. The Engel elements of a soluble group. Illinois J. Math. **3** (1959), 151–168.

An ordered pair of subsets (A,B) of a given group G shall be said to satisfy the Engel condition if to each $a\in A$

and $b \in B$ there corresponds an integer $k = k(a, b)$ such that $(a, b, \cdots, b) = 1$, this being a simple commutator with k b's. For a single element g we say that g is a left Engel element if the condition holds with $A = G$, $B = g$, and a right Engel element with $A = g$, $B = G$. We say that g is a bounded left [right] Engel element if the integer k is bounded. It has been shown by Reinhold Baer [Math. Ann. **133** (1957), 256–270; MR **19**, 248] that in a group with maximal condition the set of all left Engel elements coincides with the set of all bounded left Engel elements and forms the maximal nilpotent normal subgroup, and that the set of all right Engel elements coincides with the set of all bounded right Engel elements and forms the hypercenter of the group.

This paper shows that in a solvable group the four types of Engel elements form subgroups and an example shows that these four subgroups may be distinct.

Marshall Hall, Jr. (Pasadena, Calif.)

3484:

Thompson, John. Finite groups with fixed-point-free automorphisms of prime order. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 578–581.

A proof of the long-standing conjecture that groups with the title property are nilpotent; also that a finite group, with a maximal subgroup which is nilpotent of odd order, is soluble. Both proofs use a new criterion for the existence of a normal p-complement, the proof of which is promised in a later paper. *Graham Higman* (Oxford)

3485:

Baer, Reinhold. Closure and dispersion of finite groups. Illinois J. Math. **2** (1958), 619–640.

Denoting by $\mathfrak{r}$ a set of primes, by $P\mathfrak{r}$ the complementary set of primes, and by an $\mathfrak{r}$-group [$\mathfrak{r}$-element] a finite group [element] whose order is divisible only by primes in $\mathfrak{r}$, the author calls a group $\mathfrak{r}$-closed if its $\mathfrak{r}$-elements form a characteristic $\mathfrak{r}$-subgroup, and calls a group $P\mathfrak{r}$-homogeneous if its elements induce $P\mathfrak{r}$-automorphisms in its $P\mathfrak{r}$-subgroups. Clearly $\mathfrak{r}$-closure implies $P\mathfrak{r}$-homogeneity. It is shown that G is $\mathfrak{r}$-closed if and only if G is $P\mathfrak{r}$-homogeneous and either (1) G is $\mathfrak{r}$-separated (where $\mathfrak{r}$-separation means that the composition factors of G are either $\mathfrak{r}$-groups or $P\mathfrak{r}$-groups); or (2) $\{R, P\}$ is an $\mathfrak{r}$-p-group whenever R is a maximal $\mathfrak{r}$-subgroup of G, and P a p-Sylow subgroup of G for p not in $\mathfrak{r}$. If $\mathfrak{r}$ is a single prime p then Pp-closure and $\mathfrak{r}$-homogeneity are equivalent. Also a group G is simple if it is Pp-homogeneous but not p-closed, yet every proper subgroup or epimorphic image is p-closed. A group G is called σ-dispersed if G is $\mathfrak{a}$-closed for every σ-segment $\mathfrak{a}$ of $\mathfrak{s}$; where under a partial ordering σ of a set of primes $\mathfrak{s}$, a subset $\mathfrak{a}$ is called a σ-segment if $p \in \mathfrak{a}$ and $q\sigma p$ imply $q \in \mathfrak{a}$. A group G is called p-normal if the center ZP of any p-Sylow subgroup P of G coincides with the center ZQ of any other p-Sylow subgroup Q that contains it. A more general concept of complete p-normality of a group G is then shown to be definable by six equivalent conditions one of which is the condition: whenever K is a normal subgroup of a subgroup S of index prime to p in G, then S/K is p-normal. A number of theorems are proved that relate to these concepts of $\mathfrak{r}$-closure, $P\mathfrak{r}$-homogeneity, σ-dispersion and complete p-normality.

J. S. Frame (East Lansing, Mich.)

3486:

Suzuki, Michio. On finite groups containing an element of order four which commutes only with its powers. Illinois J. Math. **3** (1959), 255–271.

The paper proves that if a finite group G contains an element π of order 4 and if π commutes only with its powers, then either (i) G contains a normal subgroup of index 2 not containing π, or (ii) G contains an abelian normal subgroup G_0 of odd order such that G/G_0 is one of the following groups: SL(2, 3), SL(2, 5), LF(2, 7), (alternating groups) A_6 and A_7. The proof starts with showing that a 2-Sylow group of G containing π as above is generated by π and an element ρ with one of the five relations: (1) $\rho^2 = 1$, $\rho\pi\rho^{-1} = \pi^{-1}$ (dihedral group of order 8); (2) $\rho^2 = \pi^2$, $\rho\pi\rho^{-1} = \pi^{-1}$ (quaternion group); (3) $\rho^{2m} = \pi^2$ ($m \geqq 2$), $\pi\rho\pi^{-1} = \rho^{-1}$ (generalized quaternion group); (4) $\rho^{2m} = \pi^2$ ($m \geqq 2$), $\pi\rho\pi^{-1} = \rho^{-1}\pi$; (5) $\rho = 1$. If (3), (4), or (5) is the case, then (i) holds. If (2) is the case and if (i) does not hold, then (ii) holds with $G/G_0 \cong$ SL(2, 3) or SL(2, 5). In the proof, theorems of Burnside and Grün and results of the author [J. Fac. Sci. Univ. Tokyo, Sect. I **6** (1951), 259–293; Amer. J. Math. **77** (1955), 657–691; MR **13**, 909; **17**, 580] and Fowler [Thesis, Univ. of Michigan, 1951] are used. If (1) is the case, finally, (i) or (ii) with LF(2, 7), A_6 or A_7 holds. Brauer's [Proc. Nat. Acad. Sci. U.S.A. **30** (1944), 109–114; **32** (1946), 182–186; MR **6**, 34; **8**, 14] theory of blocks, in particular, the theory of correspondence between blocks of a group and those of a subgroup, and a certain analysis of algebra of conjugate classes are combined with the above-mentioned tools in the long proof for this last case consisting of further case distinctions.

T. Nakayama (Nagoya)

3487:

Brauer, R.; Suzuki, Michio; and Wall, G. E. A characterization of the one-dimensional unimodular projective groups over finite fields. Illinois J. Math. **2** (1958), 718–745.

The paper gives a group-theoretical characterization of LF(2, q) ($q \geqq 4$), the group of all one-dimensional unimodular projectivities over a finite field with q elements. Thus, it is proved that a finite group G is isomorphic to LF(2, q) with a prime power $q \geqq 4$ when the following three conditions are satisfied: (I) the order g of G is even; (II) if A and B are two cyclic subgroups of G of even orders, and if $A \cap B \neq 1$, then there is a cyclic subgroup of G which includes both A and B; (III) G coincides with its commutator subgroup. The paper first considers a group G satisfying (I), (II) and shows that the 2-Sylow group T of G is either dihedral or abelian of type $(2, \cdots, 2)$. Then these two cases are treated separately. In the former case the condition (III) is replaced by a weaker one that G has no normal subgroup of index 2, and irreducible characters χ_j of G and their restrictions to a certain cyclic subgroup are studied. The result is combined with a consideration on the algebra of conjugate classes of G, to compute the degrees of χ_j and to show that g is of the form $2h(2h+\tau)(h+\tau)$ with $\tau = \pm 1$. Further considerations on classes and characters show that $2h+\tau$ is a prime power p^n, and the normalizer of a p-Sylow group of G yields a representation of G as a doubly transitive permutation group of certain type. Finally Zassenhaus' [Abh. Math. Sem. Hamburg **11** (1936), 17–40] method is applied, to yield $G \cong$ LF(2, p^n). The proof for the case with an abelian 2-Sylow group of type $(2, \cdots, 2)$ runs rather similarly and gives $G \cong$ LF(2, 2^n). *T. Nakayama* (Nagoya)

3488:

★**Dickson, Leonard Eugene. Linear groups: With an exposition of the Galois field theory.** With an introduction by W. Magnus. Dover Publications, Inc., New York, 1958. xvi+312 pp. $1.95.

An unaltered republication of the first edition [Teubner, Leipzig, 1901] with a new introduction by Wilhelm Magnus.

3489:

Zamansky, Marc. Groupes de Riesz. C. R. Acad. Sci. Paris **248** (1959), 2933–2934.

Let E be a lattice-ordered abelian group. Let F be an ordered group and let f be an order-preserving homomorphism of E onto F. The author defines Cauchy sequences in E by means of $f(|x|)$, and claims that each such Cauchy sequence is equivalent to a monotone Cauchy sequence. {The construction given has many gaps. One suspects that some additional hypotheses are needed. For example, if F is linearly ordered and f is a lattice homomorphism, then the claim is true.} *P. F. Conrad* (New Orleans, La.)

3490:

Conrad, Paul. A correction and improvement of a theorem on ordered groups. Proc. Amer. Math. Soc. **10** (1959), 182–184.

In same Proc. **9** (1958), 382–389 [MR **21** #1340] the author had stated three theorems giving sufficient conditions for certain groups of order-preserving automorphisms of (totally) ordered groups to be capable of being ordered. The second and third theorems are now corrected by strengthening the hypothesis, and the third is at the same time extended by dropping commutativity from the hypothesis. It is shown that the new, strengthened hypotheses are really more restrictive than the original one; but the question whether the theorems in their original form are perhaps still true is not attempted. *B. H. Neumann* (Manchester)

3491:

Conrad, Paul. Non-abelian ordered groups. Pacific J. Math. **9** (1959), 25–41.

The present paper is based on an earlier paper by the same author [Proc. Amer. Math. Soc. **6** (1955), 516–528; MR **17**, 458], where definitions and notation are to be found. First the order-preserving automorphisms (o-automorphisms) of a totally ordered group (o-group) that is an o-extension of a normal convex o-subgroup are studied, and certain groups of such o-automorphisms are shown to be capable of being ordered. Central o-extensions and splitting o-extensions receive special attention. Other special situations that are examined in some detail are o-extensions of archimedean o-groups, and central extensions whose factor system is bilinear. In the final section, certain o-extensions of archimedean o-groups by archimedean o-groups (o-groups of rank 2) are constructed. Some of the extensive literature on the subject of ordered groups is noticed. *B. H. Neumann* (Manchester)

3492:

Choe, Tae Ho. The interval topology of a lattice ordered group. Kyungpook Math. J. **1** (1958), 69–74.

This paper consists of four theorems, one of which is false and one that may be false. (1) If L is a cardinal product of linearly ordered cyclic groups, then L is a topological group with respect to its interval topology if and only if L is cyclic. (2) An l-group is discrete in its interval topology if and only if it is a linearly ordered cyclic group. This is false because any linearly ordered group that contains a positive element that covers the identity is discrete in its interval topology. (3) An l-group is complete if and only if each closed interval of the form $[e, x]$ is compact in the interval topology, where e is the identity of L and $e<x$. This result is well known. (4) Let H be an l-ideal of the l-group L. Then the natural homomorphism of L onto L/H is a closed mapping with respect to the interval topologies in L and L/H. The proof of this theorem has a rather large gap in it. The author assumes that a closed set in L is the union of a finite number of sets from the subbasis. *P. F. Conrad* (New Orleans, La.)

3493:

Jakubík, Ján. Konvexe Ketten in l-Gruppen. Časopis Pěst. Mat. **84** (1959), 53–63. (Slovak and Russian summaries)

Let L be a lattice-ordered group. A subset R of L is convex if $x, y \in R$ and $x<z<y$ imply that $z \in R$. The main theorem proven is that if $R_1, R_2, \cdots, R_n$ are distinct maximal chains of elements in L that are also convex and contain 0, then each R_i is a subgroup of L and $L=R_1 \oplus R_2 \oplus \cdots \oplus R_n \oplus Q$ for some subgroup Q of L. Thus, modulo a translation, the maximal chains of L that are also convex are direct summands. If L is archimedean and for $0<r \in L$ the interval $[0, r]$ is a chain, then $L=R \oplus Q$, where R is an ordered subgroup that contains $[0, r]$. *P. F. Conrad* (New Orleans, La.)

3494:

Černikov, S. N. On the structure of groups with finite classes of conjugate elements. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 60–63. (Russian)

3495:

Morikawa, Hisasi. On the invariants of finite nilpotent groups. Osaka Math. J. **10** (1958), 53–56.

As a special case of a difficult problem concerning the birationality of the field of invariants of a group G of linear transformations, the author considers a finite nilpotent group $G=G_0$ of exponent N with a descending chain of normal subgroups $G_i \supset G_{i+1}$ such that G_i/G_{i+1} is cyclic in G/G_{i+1} and $G_{r+1}=\{e\}$. Assuming that the characteristic of the field k is coprime to N, the author calculates inductively a complete system of generators of the invariants of a representation $M(\sigma)$ that is a direct sum of representations $M_i(\lambda_i(\sigma))$ with multiplicities t_i, where $\lambda_i(\sigma)$ is the element of G/G_i corresponding to σ of G, and where M_i denotes the regular representation of G/G_i. Thus birationality of invariants is proved for this case. *J. S. Frame* (East Lansing, Mich.)

3496:

Graev, M. I. Irreducible unitary representations of the group of third order matrices conserving an indefinite Hermite form. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 966–969. (Russian)

3497:

Foulkes, H. O. The analysis of the characters of the Lie representations of the general linear group. Proc. Amer. Math. Soc. **10** (1959), 497–501.

The module of all forms of degree m in the n generators of a Lie ring has a matrix representation with character

$$\gamma_m = m^{-1}\sum_{d|m}\mu(d)S_d{}^{m/d},$$

where $\mu(d)$ is the Möbius function and S_r the power sum of the latent roots of the matrix of transformation of the generators. The problem is to express the representation as a direct sum of representations of the full linear group, or equivalently to express γ_m as a sum of S-functions in the form

$$\gamma_m = \sum \alpha_\lambda\{\lambda\}.$$

This can be done by expressing $S_d{}^{m/d}$ in terms of S-functions by the formula

$$S_d{}^{m/d} = \sum \chi^\lambda_{d^{m/d}}\{\lambda\}.$$

The author gives a method of obtaining the symmetric group character $\chi^\lambda_{d^{m/d}}$ which is equivalent to, though less direct than, the following.

Substitute in the formula

$$\sum \frac{h_\rho}{h}\chi_\rho{}^{(\lambda)}S_\rho = \{\lambda\} = |h_{\lambda_r-s+t}|$$

with S-functions associated with the series e^{x^d} for which $S_d=d$, $S_r=0$ when r is not a multiple of d, and $h_{dr}=1/r!$, $h_i=0$ when i is not a multiple of d. The only class giving a significant term is the class $(d^{m/d})$ for which $h/h_\rho=(m/d)!d^{m/d}$. A formula for $\chi^\lambda_{d^{m/d}}$ results.

The author also obtains a number of relations connecting the coefficients α_λ which are useful in the computation.

D. E. Littlewood (Bangor)

3498:

Kodama, Tetsuo; and Yamamoto, Koichi. Some properties of characters of the symmetric group. Mem. Fac. Sci. Kyusyu Univ. Ser. A **12** (1958), 104–112.

Assuming $n<2p$, the authors have considered those representations of S_n which belong to all blocks of lowest kind, i.e., whose degrees are prime to p. Such representations can be divided into classes and the number of those whose longest hook is of length $>p$ is given by $2\sum_s\tau(s)\wp(q-s)$, where $n=p+q$, $\tau(n)$ is the number of divisors of n and $\wp(n)$ is the number of partitions of n. The intrusion of $\tau(n)$ into such an expression is new and is of interest since the expression is independent of p. In this respect the formula is similar to those enumerating the ordinary and modular representations in a single block.

G. de B. Robinson (Toronto, Ont.)

3499:

★Dubreil, Paul. Quelques problèmes d'algèbre liés à la théorie des demi-groupes. Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956, pp. 29–44. Centre Belge de Recherches Mathématiques. Établissements Ceuterick, Louvain; Librairie Gauthier-Villars, Paris; 1957. 293 pp. 250 francs belges.

This paper presents a survey of some elementary topics in the theory of semi-groups. Its theme is the application of semi-group theory to group and ring structure problems. For example, the question of characterizing the multiplicative semi-group of a ring is raised and discussed. On the technical side, the author proves some results concerning the existence of sets K in a semi-group D which are minimal with respect to the property that if $a\in D$, then there exists an $x\in D$ satisfying $ax\in K$. Such subsets are related to the homomorphisms of D onto groups.

R. S. Pierce (Seattle, Wash.)

3500:

Pondělíček, Bedřich. Über eine Semigruppe der Endomorphismen auf einer einfach geordneten Menge. I. Časopis Pěst. Mat. **84** (1959), 177–182. (Czech. Russian and German summaries)

Various technical properties of semigroups of endomorphisms of a completely ordered set are shown to be equivalent.

E. Hewitt (Seattle, Wash.)

3501:

Tamura, Takayuki. Note on finite simple c-indecomposable semigroups. Proc. Japan Acad. **35** (1959), 13–15.

A statement, without proof, of three theorems (with lemmata and corollaries) of which the third is as follows: A finite simple semigroup G without zero has the property that its only commutative homomorph is of order 1 if and only if G/H has this property, where H is a certain specified normal subgroup of G.

H. A. Thurston (Vancouver, B.C.)

3502:

Sade, A. Système demosien associatif de multigroupoides avec un scalaire non singulier. Ann. Soc. Sci. Bruxelles. Sér. I **73** (1959), 231–234.

A multigroupoid $E(*)$ is a set E with a law of composition $*$ such that any ordered pair $x, y\in E$ determines a non-void subset $x*y$ of E. A scalar s is an element of E such that all products $x*s$, $s*x$ $(x\in E)$ are single elements; s is nonsingular if $x\to x*s$, $x\to s*x$ are permutations of E. Associativity, and isotopy, of multigroupoids are defined formally as with groupoids. A system of multigroupoids defined on E by laws of composition ϕ_i forming a population Φ is called demosian associative if for all x, y, z of E and ϕ_1, ϕ_2 of Φ there exist ϕ_3, ϕ_4 in Φ such that $(x\phi_1y)\phi_2z=x\phi_3(y\phi_4z)$. Assuming this, and that the multigroupoids $E(\phi_i)$ have all a common nonsingular scalar s, the author shows that they are all isotopic to one associative multigroupoid. "Isotopic" in this theorem may be replaced by "isomorphic" if each $E(\phi_i)$ has a scalar unit u_i $(x\phi_iu_i=u_i\phi_ix=x)$.

I. M. H. Etherington (Edinburgh)

3503:

Stein, Sherman K. On a construction of Hosszú. Publ. Math. Debrecen **6** (1959), 10–14.

Il s'agit de la construction de Hosszú, Colloq. Math. **5** (1957), 32–42 [MR **20** #4108]. Un automorphisme d'un groupe fini $G(\cdot)$ est dit "spécial" s'il déplace tous les éléments de G, sauf l'élément neutre. Une permutation P de l'ensemble G est "complète" [H. B. Mann, Ann. Math. Statist. **13** (1942), 418–423; MR **4**, 184; Hall et Paige, Pacific J. Math. **5** (1955), 541–549; MR **18**, 109] si l'application $x\to x\cdot(xP)$ est encore une permutation. Un automorphisme d'un groupe G est "régulier" si toutes ses puissances non identiques sont spéciales. Th. 1: Les énoncés

suivants sont équivalents: (a) W est un automorphisme spécial du groupe $G(\cdot)$; (b) la permutation $x \to x^{-1}W$ est un antiautomorphisme complet de G (un antiautomorphisme est un isomorphisme entre un groupoïde et son conjoint); (c) le groupoïde $R(\circ)$ défini par $x \circ y = xW \cdot y^{-1}W \cdot y$ est un quasigroupe autodistributif à droite et W ne déplace pas l'unité de G. Th. 2: Comme on peut trouver [Zappa, Boll. Un. Mat. Ital. (3) **12** (1957), 154–163; MR **19**, 836] des groupes finis non abéliens ayant un automorphisme spécial, la réponse à la question (3) de l'A. [Trans. Amer. Math. Soc. **85** (1957), 228–256; MR **20** #922] est négative: il existe des quasigroupes distributifs à droite et non à gauche. Th. 3: Un groupe d'ordre $4k+2$ n'a aucun automorphisme spécial [Hall et Paige, loc. cit.]. Th. 4: Si un groupe est automorphe par $x \to x^3$ il est abélien. Th. 5: Si W est un automorphisme spécial de G, d'ordre $n \neq 1$, alors pour chaque $x \in G$, $x \cdot xW \cdot xW^2 \cdots xW^{n-1} = 1$. Th. 6: Construction des séries orthogonales analogue à celle de Mann [Ann. Math. Statist. **14** (1943), 401–414; MR **5**, 169; p. 408]. Th. 7: Ordre d'un groupe et de ses automorphismes spéciaux. {Page 12, ligne 14, lire [9] au lieu de [7].}

A. Sade (Marseille)

3504:

Hosszú, Miklós. Nonsymmetric means. Publ. Math. Debrecen **6** (1959), 1–9.

Le but premier de ce papier était la construction d'un groupoïde cancellable infini, qui fût autodistributif d'un seul côté [Burstin-Mayer, J. Reine Angew. Math. **160** (1929), 111–130], et d'en conclure l'indépendance des postulats d'autodistributivité à droite et à gauche. La plupart des résultats avaient déjà été publiés par l'A. [Colloq. Math. **5** (1957), 32–42; MR **20** #4108]. Sur le corps des réels, la fonction $m(x, y) = \varphi[pf(x) + qf(y)]$, $p+q=1$, $p \neq q \neq 0$, où f est une fonction admettant la fonction inverse φ, est appelée moyenne quasi-linéaire, si $p=q$ la moyenne est dite quasiarithmétique. Th. 1: Soit $(\xi = f,\ \eta = g,\ \zeta = h)$ l'isotopie qui applique un groupe G, noté multiplicativement, sur un quasigroupe A, d'opération $(\cdot)$. Alors, pour que A soit autodistributif à droite il faut et il suffit que A soit idempotent, que fh^{-1} soit un automorphisme de G déplaçant tous les éléments non identiques, et que l'application $x \to (fh^{-1}x^{-1})x$ soit une permutation de l'ensemble G. Th. 2: Avec les mêmes notations, pour que A soit autodistributif bilatère il faut et il suffit, en plus, que G soit abélien. Th. 3: Si A est un quasigroupe distributif des deux côtés et G son image par l'isotopie inverse de (f, g, h), alors pour que G soit un groupe il faut et il suffit qu'il existe un élément $u \in A$, satisfaisant avec x, y, v quelconques à la loi d'entropie dans A, $(x \cdot y) \cdot (u \cdot v) = (x \cdot u) \cdot (y \cdot v)$; si un quasigroupe autodistributif bilatère est isotope d'un groupe, il est entropique. Th. 4: La solution cancellable et continue la plus générale de la condition d'autodistributivité bilatère est la moyenne quasilinéaire. Th. 5: Si $m(x, y)$ est une fonction croissante de x, sur le corps des réels, ayant ses dérivées partielles du premier ordre par rapport à x et à y non nulles, et si $m(x, y)$ est autodistributive d'un côté, alors m est une moyenne quasilinéaire. Plusieurs de ces propositions se laissent étendre aux systèmes demosiens de groupoïdes. [Sur ces questions, cf. Knaster, Colloq. Math. **2** (1949), 1–4; MR **12**, 395; Ryll-Nardzewski, Studia Math. **11** (1949), 31–37; MR **12**, 12; et Stein, Trans. Amer. Math. Soc. **85** (1957), 228–256; MR **20** #922.]

A. Sade (Marseille)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 3482, 3492, 3832, 3876.

3505:

Vilenkin, N. Ya. Dyadicity of the group space of bicompact commutative groups. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 79–80. (Russian)

Following P. S. Aleksandrov, a compact (here "bicompact") space is called dyadic if it is a continuous image of a cartesian product of two-point Hausdorff spaces. The question was raised by Aleksandrov whether every compact group is dyadic (as a topological space); an affirmative answer is given here for the case of commutative groups.

J. L. Tits (Brussels)

3506:

Kuz'minov, V. Alexandrov's hypothesis in the theory of topological groups. Dokl. Akad. Nauk SSSR **125** (1959), 727–729. (Russian)

Let D be the cyclic group of order two. A topological group is called dyadic if it is a homomorphic image of a topological product D^τ (for some cardinal τ). The author shows that every bicompact group is dyadic, thus answering a question of P. S. Alexandrov. For connected groups this follows from results of A. Weil [*L'intégration dans les groupes topologiques et ses applications*, 2ème éd., Hermann, Paris, 1953; p. 91; for a review of the 1940 ed., see MR **3**, 198] and Vilenkin [#3505 above], while a theorem of Mostert [Duke Math. J. **23** (1956), 57–71; MR **17**, 771] now reduces the proof to the case of 0-dimensional groups. The proof for this case is based on the following lemma. Given a space F which is the inverse limit of spaces F_α, $F = \lim_{\leftarrow} \{F_\alpha, \varphi_\alpha{}^\beta\}$, such that $\varphi_\alpha{}^\beta : F_\beta \to F_\alpha$ is onto, let G be a bicompact space with onto mappings $\pi_\alpha : G \to F_\alpha$ such that (i) $\pi_\alpha = \varphi_\alpha{}^\beta \pi_\beta$, and (ii) given $x, y \in G$, $x \neq y$, there exists an index α such that $\pi_\alpha x \neq \pi_\alpha y$. Then there is a homeomorphism $\xi : G \to F$ satisfying $\pi_\alpha = \varphi_\alpha \xi$, where φ_α is the natural projection $F \to F_\alpha$.—The theorem is now proved by taking in the 0-dimensional bicompact group G a well-ordered descending sequence of subgroups G_α with trivial intersection (and $G_0 = G$) such that G_α is a proper normal open subgroup of $G_{\alpha-1}$ or is $\bigcap_{\beta<\alpha} G_\beta$ according as α is not or is a limit number. By the lemma, G is the inverse limit of the factor spaces G/G_α, and by induction on α, G is shown to be homeomorphic to the topological product of the finite spaces $G_{\alpha-1}/G_\alpha$.

P. M. Cohn (Manchester)

3507:

Saito, Masahiko. Sur certains groupes résolubles. C. R. Acad. Sci. Paris **248** (1959), 1909–1911.

Let G be a solvable simply connected Lie group with Lie algebra $\mathfrak{G}$. The author gives necessary and sufficient conditions in terms of $\mathfrak{G}$ in order that G should possess a discrete, commutative uniform subgroup D. If these conditions are satisfied and G is of even dimension then G has a right invariant Kählerian structure J. Also D and J can be chosen in such a way that G/D is algebraic in the Kählerian structure induced by J.

S. Helgason (New York, N.Y.)

3508:

Vilenkin, N. J. The matrix elements of irreducible unitary representations of the group of real orthogonal matrices and group of Euclidean $(n-1)$-dimensional space motions. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 16–19. (Russian)

3509:

Stinespring, W. Forrest. A semi-simple matrix group is of type I. Proc. Amer. Math. Soc. **9** (1958), 965–967.

The author gives a very simple proof of the following special case of a theorem of Harish-Chandra: Any unitary representation of a connected semi-simple matrix group G is of type I. An algebra is called of type I $(\leqq n)$ if it satisfies the identities

$$[A_1, \cdots, A_r] = \sum_{i_1,\cdots,i_r} \operatorname{sgn}\begin{pmatrix}1\cdots r\\ i_1\cdots i_r\end{pmatrix} A_{i_1}\cdots A_{i_r} = 0$$

for all r for which the algebra of all $n\times n$ matrices satisfies them. Let K be a maximal compact subgroup of G and E a minimal projection in the center of the group algebra of K. Let $\mathfrak{A}$ be the group-algebra of continuous functions of compact support on G. Let n be the degree of the irreducible representation of K corresponding to E. Using a finite-dimensional faithful representation of G the author shows that the algebra $E\mathfrak{A}E$ is of type I $(\leqq n^2)$. The fact that any unitary representation of G is of type I is a direct consequence of this. *F. I. Mautner* (Paris)

3510:

Graev, M. I. Unitary representations of real simple Lie groups. Trudy Moskov. Mat. Obšč. **7** (1958), 335–389. (Russian)

Let $n=p+q$ where q and p are positive integers with $p\geqq q\geqq 1$. Let $G_{p,q}$ denote the group of all $n\times n$ complex non-singular matrices which leave invariant the form $z_1\bar z_1+\cdots+z_p\bar z_p-z_{p+1}\bar z_{p+1}-\cdots-z_{p+q}\bar z_{p+q}$. For each $r=0, 1,\cdots,q$ the author describes a series d_r of unitary representations of $G_{p,q}$ parameterized by the characters of the Abelian group $V_r\times T_{n-r}$ where V_r is an r dimensional vector group and T_{n-r} is the direct product of $n-r$ replicas of the circle group. For the case $r=0$ the representations are shown to be irreducible and for every r an explicit formula for the characters of the representations in the series d_r is given.

Let G_n denote the group of all $n\times n$ complex non-singular matrices and let K_n denote the subgroup of G_n consisting of all matrices which vanish below the main diagonal. Then there are just $q+1$ double cosets of the form $K_n y G_{p,q}$ and there is a natural one to one correspondence between them and the $q+1$ series d_r. The series d_0 corresponds to the double coset containing the identity and plays a basic role in that it is used in defining the others. Let $\mathfrak{X}$ be any character of $K_n\cap G_{p,q}\simeq T_n$, the group of all unitary diagonal matrices. Let $U^{\mathfrak{X}}$ denote the corresponding "induced" representation of $G_{p,q}$ as defined by the reviewer [Ann. of Math. (2) **55** (1952), 101–139; MR **13**, 434]. $U^{\mathfrak{X}}$ will not be irreducible but the double coset $K_n G_{p,q}$ has the structure of a complex analytic manifold and this leads to a notion of analyticity for elements of the space of $U^{\mathfrak{X}}$. The analytic elements form an invariant closed subspace which defines an irreducible subrepresentation of $U^{\mathfrak{X}}$. This subrepresentation is the general member of the series d_0.

The series d_r for $r>0$ corresponds to a double coset containing an element y which transforms the basic form into one whose matrix is

$$\begin{bmatrix}0&0&s\\0&\sigma&0\\s&0&0\end{bmatrix},$$

where s is an $r\times r$ matrix of the form

$$\begin{bmatrix}0&\cdot&\cdot&\cdots&0&1\\0&\cdot&\cdot&\cdots&1&0\\\cdot&\cdot&\cdot&\cdots&\cdot&\cdot\\1&0&0&\cdots&\cdot&0\end{bmatrix}$$

and σ is a diagonal matrix with $p-r$ entries equal to 1 and $q-r$ entries equal to -1. When $G_{p,q}$ is transformed by the inner automorphism defined by y it becomes a matrix group containing as a subgroup the set of all matrices of the form

$$\begin{bmatrix}g_{-1}&0&0\\0&g_0&0\\0&0&g_1\end{bmatrix},$$

where g_0 is a member of $G_{p-r,q-r}$, g_1 is an arbitrary non-singular $r\times r$ matrix and g_{-1} is the image of g_1 by a certain automorphism of the group of non-singular $r\times r$ matrices. Transforming back we define a certain subgroup H_y of $G_{p,q}$ which is isomorphic to the direct product of G_r and $G_{p-r,q-r}$. Let T be an arbitrary member of the principal series of irreducible representations of G_r as defined by Gel'fand and Naĭmark [*Unitarnye predstavleniya klassičeskih grupp*, Trudy Mat. Inst. Steklov, vol. 36, Izdat. Akad. Nauk SSSR, Moscow, 1950; MR **13**, 722]. Let T' be an arbitrary member of the series d_0 for $G_{p-r,q-r}$. Then the Kronecker product of T and T' defines an irreducible unitary representation of H_y. The corresponding induced representation of $G_{p,q}$ is the general member of the series d_r. These descriptions of d_0 and d_r are slight reformulations of those of the author who does not use the notion of induced representation explicitly.

The author announced most of his results in two notes [Dokl. Akad. Nauk SSSR **98** (1954), 517–520; **103** (1955), 357–360; MR **16**, 567; **17**, 644]. Generalizations to arbitrary semi-simple Lie groups have been discussed in detail by Harish-Chandra in a series of notes and papers [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 1076–1077, 1078–1080; **41** (1955), 314–317; Amer. J. Math. **77** (1955), 743–777; **78** (1956), 1–41, 564–628; MR **16**, 334; **17**, 60, 282; **18**, 490]. *G. W. Mackey* (Cambridge, Mass.)

3511:

Hochschild, G.; and Mostow, G. D. Representations and representative functions of Lie groups. II. Ann. of Math. (2) **68** (1958), 295–313.

This paper is a continuation of part I, hereafter referred to as RFGI [see same Ann. **66** (1957), 495–542; MR **20** #5248]. We continue here with some of the notation of that review. Let G be a Lie group, $R(G)$ the algebra of representative functions on G, A the group of proper automorphisms of $R(G)$, G^+ the universal complexification of G, and let t, t^+ be, respectively, the natural homomorphisms of G and G^+ into A. Also let γ be the homomorphism of G into G^+ implicitly defined in RFGI. The main results of RFGI had to do with a type of group referred to

as a Tannaka group in our review of RFGI. Here G^+ and $R(G)$ are investigated under more general circumstances.

An early result settles the question of the existence of a locally faithful representation of G in the following way. Let G' be the commutator subgroup of G, S a maximal semisimple analytic subgroup of G, and T the radical of G'. Then G has a locally faithful representation if and only if the following conditions are satisfied: (1) G' is closed in G, (2) T is simply connected, and (3) $S \cap T = (1)$. A corollary then asserts that if G is solvable, connected, the existence of a locally faithful representation implies the existence of a faithful representation.

Observe that $\mathrm{Hom}(G, C)$ and $\mathrm{Hom}(G, C^*)$ are subsets of $R(G)$. Let exp denote the map from $\mathrm{Hom}(G, C)$ to $\mathrm{Hom}(G, C^*)$ given by exponentiation. Also let Q denote the image of exp. Now for any $\alpha \in A$ and any $h \in R(G)$ put $\alpha'(h) = \alpha(h)(1)$. The following characterization of the subgroup $t^+(G^+)$ of A is then given. Let $\alpha \in A$. Then $\alpha \in t^+(G^+)$ if and only if $\alpha'(\exp h) = \exp(\alpha'(h))$ for all $h \in \mathrm{Hom}(G, C)$.

Under certain assumptions the following result on the structure of $R(G)$ is obtained. Let P be the kernel of t. Then the following conditions are equivalent. (1) The factor group of the radical of G/P modulo its maximal nilpotent normal analytic subgroup is compact. (2) There is a finitely generated translation and complex conjugation stable subalgebra U of $R(G)$ such that $C + \mathrm{Hom}(G, C) \subset U$, $R(G) = U[Q]$ and the elements of Q are free over U. (3) G^+ is isomorphic with an algebraic group of complex linear transformations of a space $C \otimes V$ where V is a real vector space stable under $\gamma(G)$.

It is also proved that for a group satisfying any one of the (equivalent) conditions of the theorem above the structure of A is given by $A = W \times t^+(G^+)$ where W is isomorphic to $\mathrm{Hom}(Q, C^*)$. *B. Kostant* (Berkeley, Calif.)

3512:

★**Braconnier, Jean. Sur les groupes de Lie compacts opérant dans une variété compacte.** Séminaire Bourbaki; 10e année: 1957/1958. Textes des conférences; Exposés 152 à 168; 2e éd. corrigée, Exposé 163, 12 pp. Secrétariat Mathématique, Paris, 1958. 189 pp. (mimeographed)

This paper is an exposition of some recent results on topological transformation groups. Let G be a compact Lie group acting on a space X. If X is a compact manifold, then there are, up to conjugacy classes, only a finite number of isotropy subgroups. This was proved by Floyd [Ann. of Math. (2) **65** (1957), 505–512; MR **19**, 292] for the abelian case and by Mostow [ibid., 513–516; MR **19**, 292] in the general case. There is also a discussion of equivariant imbeddings (Mostow's work; see citation below) and of the slice and its existence. A slice is a generalization of a local cross-section and always exists. Intuitively it can be described, for X a manifold, as a small cell orthogonal to an orbit at a point. The best proof of its existence is by Mostow [ibid., 432–446; MR **19**, 291] and another proof was given by Montgomery and Yang [ibid., 108–116; MR **19**, 291]. *D. Montgomery* (Princeton, N.J.)

3513:

Nagano, Tadashi. Transformation groups with $(n-1)$-dimensional orbits on non-compact manifolds. Nagoya Math. J. **14** (1959), 25–38.

The author proves the following theorem. Let G be a connected isometry group of a non-compact n-dimensional Riemannian manifold M with an $(n-1)$-dimensional compact orbit. Then M has a fibre bundle structure such that (1) the base space is a compact G-orbit, (2) the associated principal bundle is $(G, G/H, H)$ where H is the isotropy subgroup of G at a point b of B, (3) the fibre E (containing b) is a submanifold of M which is C^1-diffeomorphic to the euclidean space of dimension $n - \dim B$, (4) the structure group H acts on E as a linear group in terms of some coordinate system of E, and finally (5) if $\dim B < n-1$ then H acts transitively on the (unit) sphere with center $b = E \cap B$ in the tangent space to E at b. Several consequences of the theorem are also given. *C. T. Yang* (Philadelphia, Pa.)

3514:

Karube, Takashi. On the local cross-sections in locally compact groups. J. Math. Soc. Japan **10** (1958), 343–347.

Suppose G is a locally compact group and H is a subgroup of G. Let p be the natural projection of G onto G/H. A local cross-section for H in G is a continuous map f of an open set in G/H into G such that pf is the identity map. Using the structure theory for locally compact groups, this paper proves that cross-sections exist in the following three cases: (1) G is finite-dimensional; (2) H does not have arbitrarily small subgroups; (3) all sufficiently small subgroups of G are in H. The author shows that cross-sections need not always exist by the following example: Let G be the product of infinitely many circle groups and let H be the product of their two-element subgroups. There can be no cross-section for H in G because G is locally-connected and H is not, whence G is not even locally homeomorphic to $H \times G/H$. *A. M. Gleason* (Cambridge, Mass.)

3515:

Montgomery, Deane. Groups on R^n or S^n. Michigan Math. J. **6** (1959), 123–130.

Let G be a compact connected Lie group acting differentiably on $M = R^n$ or S^n, r the highest dimension of any orbit, B^* the collection of orbits of dimension less than r, U^* the collection of non-singular orbits, $D^* = M^* - (U^* \cup B^*)$ the collection of exceptional orbits of highest dimension, topologized by the identification topology in the orbit space M^*, and π the natural projection $\pi\colon M \to M^*$. Then $U^* \cup D^*$ is simply connected. If $\dim \pi^{-1}(D^*) \leq n-3$, then U^* is simply connected. In any event, U^* is orientable. Moreover, every r-dimensional orbit is orientable. If $n-r$ is odd, there can be no isolated points of D^*. *P. S. Mostert* (New Orleans, La.)

3516:

Berger, Marcel. Les espaces symétriques noncompacts. Ann. Sci. École Norm. Sup. (3) **74** (1957), 85–177.

In the classical (Cartan's) theory of homogeneous Riemannian symmetric spaces one deals with G/H, where G is a connected Lie group, H is a closed subgroup of G such that: (1) $H_0 \subseteq H \subseteq H_\Sigma$ where Σ is a continuous automorphism of order 2 of G, H_Σ is the set of fixed elements of Σ and H_0 is the identity component of H; and (2) if σ is the involutory automorphism of $\mathfrak{g}$, the Lie algebra of G, induced by Σ, and $\mathfrak{h}$, $\mathfrak{m}$ are, respectively, the eigenspaces of

σ belonging to the eigenvalues 1 and -1, then the isotropy representation (ad H, $\mathfrak{m}$) of H on $\mathfrak{m}$ leaves a positive definite bilinear form on $\mathfrak{m}$ invariant. In this paper the author investigates the situation without assuming (2).

The local structure theory is reduced to finding all possible subalgebras $\mathfrak{h}$ (defined by (2)) when $\mathfrak{g}$ is simple. This problem is simplified in the following way. Let $\mathfrak{g}_1$ be the Lie algebra of the maximal compact subgroup of the adjoint group of $\mathfrak{g}$. Let τ be the involutory automorphism of $\mathfrak{g}$ whose fixed set is $\mathfrak{g}_1$. Let $\mathfrak{g}_{-1}$ (the notation here is slightly different from the author's) be the eigenspace for the eigenvalue -1 of τ. It is then observed that it is enough to assume that σ commutes with τ. This defines a direct sum decomposition $\mathfrak{g}=\mathfrak{g}_{1,1}+\mathfrak{g}_{1,-1}+\mathfrak{g}_{-1,1}+\mathfrak{g}_{-1,-1}$ where the four given spaces are eigenspaces of both τ and σ belonging to the respective eigenvalues given by the subscripts. Thus $\mathfrak{g}_1=\mathfrak{g}_{1,1}+\mathfrak{g}_{1,-1}$ and $\mathfrak{h}=\mathfrak{g}_{1,1}+\mathfrak{g}_{-1,1}$. Clearly $\mathfrak{g}_1/\mathfrak{g}_{1,1}$ defines the local structure of a compact symmetric space and hence falls within the province of the classical theory. On the other hand it is shown that up to an equivalence, if $\mathfrak{g}$ is not complex, the mapping $\mathfrak{g}/\mathfrak{h}\rightarrow\mathfrak{g}_1/\mathfrak{g}_{1,1}$ is 2 to 1. This essentially reduces the classification problem to the determination of which of the involutory automorphisms of $\mathfrak{g}_1$ (all of which are known) extend to involutory automorphisms of $\mathfrak{g}$. Much of the paper is devoted to a case by case treatment leading eventually to the list of all possible $\mathfrak{h}$ for all possible simple $\mathfrak{g}$. In all these cases the infinitesimal isotropy representation (ad $\mathfrak{h}$, $\mathfrak{m}$) is also given. It is shown that, unlike the classical case, even if $\mathfrak{g}$ is simple, (ad $\mathfrak{h}$, $\mathfrak{m}$) may be reducible. Also having (ad $\mathfrak{h}$, $\mathfrak{m}$) irreducible but not complex irreducible does not necessarily imply that $\mathfrak{h}$ has a non-trivial center.

Going over to the case of groups it is shown using a lemma of Mostow's that G/H, when G is semisimple, is a fibre space over a compact symmetric space with Euclidean space as fibre.

The case when G is non-semisimple is also treated.

B. Kostant (Berkeley, Calif.)

3517:

Wall, G. E. The structure of a unitary factor group. Inst. Hautes Études Sci. Publ. Math. no. 1, 23 pp. (1959).

The author obtains, for unitary groups, an analogue to Dieudonné's theory [Bull. Soc. Math. France **71** (1943), 27–45; MR **7**, 3] of noncommutative determinants.

Let D be a division ring with an involutory antiautomorphism $\lambda\rightarrow\bar\lambda$, V a finite-dimensional vector space over D, and $f=(x,y)$ a map: $V\times V\rightarrow D$ which is linear and semi-linear with respect to y and x respectively. Assume that f is non-degenerate and skew-Hermitian and that, for every x, (x,x) is of the form $\lambda-\bar\lambda$, $\lambda\in D$. Let $U(f)$ be the group of all unitary transformations of V, and $T(f)$ the group generated by all unitary transvections. Assuming that V contains non-zero isotropic vectors, the author proves that, if $D\neq \mathrm{GF}(4)$,

$$U(f)/T(f)\cong\Delta/\Sigma[\Delta,\Omega],$$

where Δ is the multiplicative group of D, Σ the subgroup generated by symmetric elements ($\bar\lambda=\lambda$), and Ω the subgroup generated by (x,x) for x orthogonal to a hyperbolic plane. If, moreover, $T(f)$ is projectively a simple group (which is true except for some half-dozen cases) and $n\geq 3$, then $T(f)$ is the commutator group of $U(f)$.

R. Ree (New York, N.Y.)

MISCELLANEOUS TOPOLOGICAL ALGEBRA

3518:

Hosszú, Miklós. Functional equations and algebraic methods in the theory of geometric objects. Publ. Math. Debrecen **5** (1958), 294–329.

Given a space E with a semi-group S of (left) transformations acting on it; then a function $x(P)$ from E to a space X on which S acts as a (right) transformation semi-group is called a "geometric object" provided $x(\alpha P)=(x(P))\alpha$ for every $\alpha\in S$, $P\in E$. The purpose of this paper is the classification of geometric objects, that is, the classification of the spaces X on which S acts as a right-transformation semi-group. This problem is solved in case S is a group G or, more generally, in case S is assumed to be a continuous semi-group acting on X containing a group G which is dense in S.

The space X is first split into orbits under G (under the assumption that the identity of G acts as identity transformation) and these orbits are isomorphic to the right cosets of a subgroup of G. A detailed study of examples in which both E and X are vector spaces is given. The final section is devoted to the study of certain finite-dimensional algebras with certain semi-groups of endomorphisms.

E. G. Straus (Los Angeles, Calif.)

3519:

Mikulík, Miloslav. A note on lattices with distance functions. Časopis Pěst. Mat. **84** (1959), 1–6. (Czech. Russian and English summaries)

Es sei S ein Verband, der gleichzeitig ein metrischer Raum mit der Metrik ρ ist. Der Verf. untersucht die Bedingungen, unter welchen S die folgende Eigenschaft (E) hat: In S ist die metrische Konvergenz mit der o-Konvergenz identisch. Ist F eine Folge von Elementen x_i $(i=1,2,\cdots)$ der Menge S, so setzen wir

$$D_n(F)=\rho\Big(\bigvee_{i=n}^{\infty}x_i,\quad\bigwedge_{i=n}^{\infty}x_i\Big),$$

wenn das Supremum und Infimum existiert. Es sei $d(F)$ der Durchmesser der aus allen x_i gebildeten Menge. Wenn die Metrik ρ eine gewisse Bedingung erfüllt, welche schwächer als die Kompaktheit ist, und wenn für jede Folge F von Elementen der Menge S, für die $D_1(F)$ definiert ist, die Gleichung $D_1(F)=d(F)$ richtig ist, so hat S die Eigenschaft (E). Ist S ein σ-vollständiger Verband, der ein kompakter metrischer Raum ist, so hat S genau dann die Eigenschaft (E), wenn für jede metrisch konvergente Folge F von Elementen der Menge S die Gleichung $\lim_{n\rightarrow\infty}D_n(F)=0$ richtig ist.

M. Novotný (Brno)

FUNCTIONS OF REAL VARIABLES

See also 3576, 3717, 3730, 4052.

3520:

Froda, Alexandre. Chaînes dirigées—pour l'étude de propriétés transitives "à distance", des fonctions réelles. Math. Nachr. **19** (1958), 1–28.

The author extends, using directed chains, the properties "at a distance" of real functions on Euclidean

n-space E^n, $n>1$, as discussed in earlier papers [for example, Bull. Sci. Math. (2) **21** (1957), 175–180; MR **20** #3940]. A para-neighborhood of $P \in E^n$ is the boundary of a starlike neighborhood of P subject to certain restrictions. Interest centers about properties of a function f in terms of its behavior on the para-neighborhood.

M. E. Shanks (Lafayette, Ind.)

3521:

Burnengo, Giuseppe. Sugli estremi relativi della funzione $y=f(x_1, x_2, \cdots, x_n)$. Archimede **11** (1959), 97–104.

In questa nota mi propongo di stabilire le condizioni necessarie e quelle sufficienti per un estremo della funzione $y=f(x_1, x_2, \cdots, x_n)$ con considerazioni di indole geometrica e intuitiva. *Author's summary*

3522:

Benneton, Gaston. Sur la formule de Green et les conditions suffisantes d'analyticité. C. R. Acad. Sci. Paris **248** (1959), 2548–2549.

The formula of Green is established in the plane without imposing conditions on the partial derivatives, but only on their difference. Let C be a continuous closed rectifiable curve in the plane which determines a closed region D. Let $P(x, y)$, $Q(x, y)$ and $\varphi(x, y) = \partial Q/\partial x - \partial P/\partial y$ be continuous on D. Then

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy.$$

Thus the formula of Green holds when neither of the derivatives $\partial Q/\partial x$, $\partial P/\partial y$ is necessarily either bounded or continuous. By a change of variable in this formula, which amounts to introducing a system of curvilinear coordinates, the following result is obtained.

V is a vector function continuous on D, **V*** the vector resulting from a rotation of $-\pi/2$ of **V**. If at each point axes MX and MY are taken, rectangular or not, varying continuously with M, and if P^* and Q^* are the components of **V*** on these axes, then

$$\int_C \mathbf{V} dM = \iint_D \left[\frac{\partial P^*}{\partial X} + \frac{\partial Q^*}{\partial Y}\right]$$

under the conditions that the two partial derivatives remain finite and that their sum is continuous.

The first formula is directly applicable to functions of a complex variable $f(z)$. If the conditions of Cauchy hold almost everywhere, then the difference of the resulting partial derivatives is zero almost everywhere and the Cauchy-Goursat theorem follows. The second formula leads to a more general result. In order that a continuous function be analytic on D it is sufficient that the derivatives of this function taken in two fixed directions are equal on D. *R. L. Jeffery* (Kingston, Ont.)

3523:

Halperin, Israel. Discontinuous functions with the Darboux property. Canad. Math. Bull. **2** (1959), 111–118.

Expository review of examples of functions which possess the Darboux intermediate-value property but lack various continuity, integrability, and measurability properties. *T. A. Botts* (Charlottesville, Va.)

3524:

Musielak, J.; and Orlicz, W. On generalized variations. I. Studia Math. **18** (1959), 11–41.

The M-variation of a function $x(t)$ is defined as

$$\sup \sum_{i=1}^m M[|x(t_i) - x(t_{i-1})|],$$

the supremum being taken over all partitions. Subjecting the function M to various appropriate conditions of growth, convexity, etc., the authors give results on related concepts of absolute continuity, convergence in variation, a generalised Helly-Bray theorem, etc.

J. T. Schwartz (New York, N.Y.)

3525:

Shamir, E. On the existence of continuous nowhere differentiable functions. Bull. Res. Council Israel. Sect. F **7F** (1957/58), 77–82.

The author discusses the existence or non-existence of derivatives of a function $f(x)$ over a family of sets S_x, and shows that, if the family has a certain property which he calls "geometric (β)", there exist functions which are continuous for all x but are not differentiable over S_x for any x. He shows that there exist families S_x for which this construction is impossible.

U. S. Haslam-Jones (Oxford)

3526:

Tietz, Horst. Ein Satz über nicht-positive Funktionen und seine Anwendung in der Theorie der Differentialgleichungen. Jber. Deutsch. Math. Verein. **61** (1958), Abt. 1, 93–96.

It is shown that if a summable function $f(x)$ satisfies: $f(x) \leq L|\int_c^x f(t)dt|$ for some c and all x on the interval (a, b), where $L \geq 0$ is a constant, then $f(x) \leq 0$ for all x on (a, b). The foregoing result is used to establish the usual uniqueness and continuity properties of solutions of the differential equation $y' = F(x, y)$ through a point (α, β) when $F(x, y)$ satisfies a Lipschitz condition and the initial values vary as parameters.

W. M. Whyburn (Chapel Hill, N.C.)

3527:

Murta, Manuel. A theorem on homogeneous functions. Rev. Fac. Ci. Univ. Coimbra **27** (1958), 22–23. (Portuguese)

"Every p-homogeneous function of class C^p in a region containing the origin is a polynomial."

3528:

Olkin, Ingram. On inequalities of Szegö and Bellman. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 230–231; addendum, 1553.

It is proved that if $1 \geq w_1 \geq w_2 \geq \cdots \geq w_m \geq 0$, $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$, and f is a continuous convex function in $[0, a_1]$, then

$$\left[1 - \sum_1^m (-1)^{j-1} w_j\right] f(0) + \sum_1^m (-1)^{j-1} w_j f(a_j) \geq f\left[\sum_1^m (-1)^{j-1} w_j a_j\right].$$

This has also been proved by H. D. Brunk [Proc. Amer. Math. Soc. **7** (1956), 817–824; MR **18**, 391; Corollary 3], but the present proof is more elementary.

F. F. Bonsall (Newcastle-upon-Tyne)

MEASURE AND INTEGRATION

See also 3336, 3696, 3820, 3917, 4061.

3529:

Marcus, S. Sur une théorie du type Lebesgue pour l'intégrale de Riemann. Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 333–369. (Romanian. Russian and French summaries)

A fairly extensive discussion, principally along methodological lines, of the Riemann integral. The author uses vertical partitions which avoid a denumerable (at most) exceptional set. This fact is exploited in the treatment. He points out the significance and advantages for the theories of Riemann and Lebesgue of the horizontal and vertical partitions. The section headings indicate the scope of the discussion: Jordan measure; Jordan measurable functions; Riemann integral of a bounded function; a second definition of the notion of "Riemann integrable function"; a new definition of integrability; the Riemann integral of a non-negative Jordan measurable function; Riemann summable functions of variable sign.

E. R. Lorch (New York, N.Y.)

3530:

Godefroid, Michel. Calcul d'une intégrale double par deux intégrations simples successives (intégrale de Riemann). Enseignement Math. (2) **5**(1959), 58–60.

Let the function $f(x, y)$ be bounded and Riemann integrable over the square $C=[-a, a]\times[-a, a]$ in the x, y plane. It is shown that for every function $F(x)$ on the interval $[-a, a]$ for which

$$\underline{\int_{-a}^{a}} f(x, y)dy \leq F(x) \leq \overline{\int_{-a}^{a}} f(x, y)dx$$

($\overline{\int}$ and $\underline{\int}$ denoting upper and lower Darboux integrals), it is the case that $\int_{-a}^{a} F(x)dx$ exists and equals $\iint_C f(x, y)dxdy$.

T. A. Botts (Charlottesville, Va.)

3531:

Vinti, Calogero. L'integrale di Weierstrass. Ist. Lombardo Accad. Sci. Lett. Rend. A **92** (1957/58), 423–434.

The author derives the equality of the integral of Weierstrass and a limit which has been used by Baiada and others [see, e.g., E. Baiada and G. Tripiciano, Rend. Circ. Mat. Palermo **6** (1957), 263–270; MR **20** #4201], under weak assumptions on the integrand.

L. M. Graves (Chicago, Ill.)

3532:

Nakanishi, Shizu. Sur la dérivation de l'intégrale (E. R.) indéfinie. I, II. Proc. Japan Acad. **34** (1958), 199–204, 257–262.

L'A. nomme intégrale (E. R.) l'intégrale introduite par K. Kunugi [mêmes Proc. **32** (1956), 215–220; MR **18**, 567]. Il étudie la dérivation de cette intégrale.

A. Appert (Angers)

3533:

Heider, L. J. A representation theory for measures on Boolean algebras. Michigan Math. J. **5** (1958), 213–221.

Let B be a Boolean algebra. We shall write: $\phi\in M$ if ϕ is a measure on B, that is, if $\phi(a\vee b)=\phi(a)+\phi(b)$ for $a\wedge b=0$, a, $b\in B$, $\sup|\phi(a)|<\infty$; $\phi\in M_c$ if $\phi\in M$ is σ-additive, that is, if $\phi(a_n)\to 0$ for $a_n\searrow 0$; $\phi\in M_{pf}^+$ if $\phi\geq 0$ is purely finitely additive, that is, if $0\leqq\psi\leqq\phi$, $\psi\in M_c$ implies $\psi=0$; $\phi\in M_{pf}$ if $\phi=\phi^+-\phi^-$, $\phi^+\in M_{pf}^+$, $\phi^-\in M_{pf}^+$; and $\phi\in M_f$ if $\phi\in M-M_c$. Let X_0 be the Stone-representation space of B. The author proves the following results. The $\phi\in M$ are in 1-1 correspondence $\phi\leftrightarrow\bar{\phi}$ to the Baire measures $\bar{\phi}$ on X_0. $\phi\in M_c$ if and only if $\bar{\phi}$ vanishes on all sets of the 1st category, $\phi\in M_{pf}$ if $\bar{\phi}$ vanishes outside of a set of the 1st category. Each of the sets M, M_c and M_{pf} constitutes an abstract (L)-space. If B is a σ-algebra and each family of disjoint elements is denumerable, then there exists a $b_0\in B$ and a $\phi\in M_c$ such that $\phi(b)>0$ if $0<b<b_0$, and $\psi(b)=0$ for any $\psi\in M_c$ and any $b<$complement of b_0. Applying these results to the case where B is a σ-field of subsets of a set Y, the author obtains a reformulation of the work of Yosida and Hewitt [Trans. Amer. Math. Soc. **72** (1952), 46–66; MR **13**, 543].

M. Cotlar (Buenos Aires)

3534:

Cesari, L.; and Turner, L. H. Surface integral and Radon-Nikodym derivatives. Rend. Circ. Mat. Palermo (2) **7** (1958), 143–154.

Die Arbeit enthält die Beweise vorher [Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 42–43; MR **20** #954] angekündigter Ergebnisse der Verfasser.

K. Krickeberg (Aarhus)

FUNCTIONS OF A COMPLEX VARIABLE

See also 3383, 3522, 3785, 3934.

3535:

Plunkett, Robert L. A topological proof of the continuity of the derivative of a function of a complex variable. Bull. Amer. Math. Soc. **65** (1959), 1–4.

Let $f(z)$ be a nonconstant complex-valued function which has a finite derivative at every point of a domain D in the complex plane. With the help of two theorems proved in *Topological analysis* [G. T. Whyburn, Princeton Univ. Press, 1958; MR **20** #6081], the author proves the continuity of $f'(z)$ and thus the analyticity of $f(z)$. The best previous result is due to Whyburn [p. 89, op. cit.] in which the fact that $f'(z)$ has closed and scattered point inverses was obtained.

C. J. Titus (Berkeley, Calif.)

3536:

Faber, Karl. Begründung der Potential- and Funktionentheorie mit Einschluss gewisser Erweiterungen. Jber. Deutsch. Math. Verein. **61** (1958), Abt. 1, 32–56.

Analytic functions have been characterized both by differential and integral properties. The present paper is addressed to the question of which is the more natural characterization. To this end the author discusses the characterization by line integral properties of harmonic functions, of functions satisfying $u_{xx}-u_{yy}=0$, and of those satisfying $u_{xx}=0$, and is eventually led "naturally" to a consideration of analytic functions of a complex variable.

J. W. Green (Los Angeles, Calif.)

3537:

Dundučenko, L. O.; and Kas'yanyuk, S. A. On two classes of functions regular in n-connected circular regions. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 468–472. (Ukrainian. Russian and English summaries)

The authors present and study two very general classes $\Gamma_0(K_n)$ and $\Gamma_*(K_n)$ regular in n-connected circular regions of functions, between which there exists a relationship of the Alexander type. The structural formulae of these classes are determined and a number of exact estimates are obtained. Concretization of the parameters of these classes makes it possible to single out of them most classes of functions which are n-connected analogues of the respective classes of functions, adequately studied in the circle and annulus. *Author's summary*

3538:

Loewner, Charles. On the conformal capacity in space. J. Math. Mech. **8** (1959), 411–414.

Ein quasikonformer Homöomorphismus der euklidischen Ebene in sich ist notwendig eine Abbildung auf. Wegen des Riemannschen Abbildungssatzes genügt es zum Beweise dieses Satzes zu zeigen, dass es keine eineindeutige quasikonforme Abbildung der Ebene in die Kreisscheibe gibt. Letzteres lässt sich leicht auf höhere Dimensionen verallgemeinern; dagegen erfordert die Uebertragung des ersten Satzes neue Mittel. Diese wurden vom Verf. gegeben durch den Begriff der konformen Kapazität. Es sei D ein Gebiet des 3-dim. euklidischen Raumes mit zwei Randkomponenten C_1 und C_2. Das Infimum des Integrals $\int_D |\nabla u|^3 d\omega$ für alle in D stetig differenzierbaren Funktionen u, die auf C_1 verschwinden und auf C_2 gleich 1 sind, ist eine konforme Invariante und wird deshalb vom Verf. die konforme Kapazität des Gebietes D genannt. Er beweist: Diese Kapazität ist dann und nur dann positiv, wenn keine der Randkomponenten C_1 und C_2 zu einem Punkt degeneriert.

Unter K-quasikonformer Abbildung im R_3 wird hier eine solche stetig differenzierbare Abbildung verstanden, die eine infinitesimale Kugel in ein Ellipsoid überführt, deren grösste Achse höchstens K-mal die kleinste Achse ist. Ist D' ein K-quasikonformes Bild des Gebietes D und sind Γ' und Γ ihre konformen Kapazitäten, so gilt (analog zur Ebene) $K^2\Gamma \geqq \Gamma'$. Daraus folgt dann: Ein quasikonformer Homöomorphismus des euklidischen Raumes R_3 in sich ergibt als Bild notwendig den ganzen Raum R_3. Alle Betrachtungen lassen sich auf Räume der Dimension $n>3$ übertragen. *A. Pfluger* (Zürich)

3539:

Motzkin, T. S.; and Walsh, J. L. Location of zeros of infrapolynomials. Compositio Math. 14, 50–70 (1959).

This article sheds new light on the subject of infrapolynomials by introducing and developing the concept of a "substar". A substar is defined as a closed point set E for which there exists a non-constant rational function $f(z)=q(z)/p(z)$, in which the polynomials are $q(z)=z^n+a_1z^{n-1}+\cdots$ and $p(z)=z^n+b_1z^{n-1}+\cdots$ and for which $|f(z)|\leqq 1$ on E. Since at infinity the locus $|f(z)|=1$ is the boundary of m curvilinear sectors in which $|f(z)|<1$ and $|f(z)|>1$ alternately, a part F may be found in the complement of E such that $E+F$ is not a substar. This leads to the fact that if polynomial $p(z)\neq 0$ on F and at isolated points if any, of the boundary B of F, then $p(z)$ is an infrapolynomial on E. A consequence is that if E is a bounded non-collinear set that contains the boundary of its convex hull H, the set of nth degree infrapolynomials on E is the set of polynomials having all their zeros on H. A detailed study is made of the cases that E is a straight line, a point plus a straight line, a point plus a strip, or set of circular disks. *M. Marden* (Milwaukee, Wis.)

3540:

Heigl, Franz. Einige Schranken für die Absolutbeträge der Wurzeln algebraischer Gleichungen. Monatsh. Math. **63** (1959), 287–297.

Let the equation be $z^n+a_1z^{n-1}+\cdots+a_n=0$ with a_μ real or complex. Set $A_\mu=|a_\mu|$ $(1\leqq\mu\leqq n)$, $A_0=1$, $A_\mu=0$ if $\mu<0$ or $\mu>n$. If ζ denotes any root, then

$$|\zeta| < \max\left[\tfrac{1}{2}A_{\nu-1}+(1+A_\nu+\tfrac{1}{4}A_{\nu-1}^2)^{1/2}\right],$$

where the maximum is taken over $\nu=2, 4, 6, \cdots$. Three more bounds are derived by the same method. They depend on a number of parameters selectable at will within certain limits. Special choices of the parameters lead to known results of T. Anghelutza [Boll. Un. Mat. Ital. **12** (1933), 284–288], M. Fujiwara [Tôhoku Math. J. **10** (1916), 167–171], E. C. Westerfield [Amer. Math. Monthly **40** (1933), 18–23]. A bound due to Montel [Comment. Math. Helv. **7** (1935), 178–200] is also rederived.

Walter Gautschi (Oak Ridge, Tenn.)

3541:

Parodi, Maurice. Sur la localisation dans le plan complexe des racines des équations abéliennes. C. R. Acad. Sci. Paris **248** (1959), 2153–2154.

The equation

$$(*) \qquad x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n = 0$$

is said to be simple abelian if any root x_1 may be expressed rationally as $x_1=\theta(x_2)$ in terms of some other root x_2. Equation (*) remains invariant under the transformations $x=\theta(y)$, $x=\theta(\theta(y))=\theta^2(y), \cdots$. Hence, from previous results that the roots of (*) satisfy conditions D: $|x|\leqq 1$, $|x+a_1|\leqq\sigma$, where $\sigma=\sum_2^n |a_k|$, it follows that the roots satisfy D_k: $|\theta^{-k}(x)|\leqq 1$, $|\theta^{-k}(x)+a_1|\leqq\sigma$, $k=1, 2, \cdots, p$, where θ^{-k} is the inverse of θ^k. Hence the roots of (*) lie in the intersection of the domains $D, D_1, D_2, \cdots, D_p$.

M. Marden (Milwaukee, Wis.)

3542:

Lech, Christer. A metric result about the zeros of a complex polynomial ideal. Ark. Mat. **3** (1958), 543–554.

L'auteur, motivé par la caractérisation des systèmes hypo-elliptiques [voir L. Hörmander, #3673], montre un théorème intéressant que voici: Soit $K[x]=K[x_1, \cdots, x_n]$ un anneau de polynomes à coefficients complexes (K = corps des nombres complexes); soit $\mathfrak{a}$ un idéal de $K[x]$; désignons

$$d(\gamma; f) = \inf_{\gamma'}\{|\gamma-\gamma'|;\ \gamma'\in K^n, f(\gamma') = 0\},$$

$$d(\gamma; \mathfrak{a}) = \inf_{\gamma'}\{|\gamma-\gamma'|;\ \gamma'\in K^n, f(\gamma') = 0 \text{ pour tout } f\in\mathfrak{a}\}.$$

Alors, il existe un polynome $f\in\mathfrak{a}$ et une constante positive c tel que, pour tout α réel, on a $d(\alpha; f)\geqq c\,d(\alpha; \mathfrak{a})$.

L'A. le démontre par des outils purement algébriques.

S. Mizohata (Kyoto)

3543:

Ahmad, Salah. Sur la probabilité pour qu'une série entière à coefficients aléatoires puisse être prolongée. C. R. Acad. Sci. Paris **248** (1959), 2160–2161.

The present note is a continuation of an earlier paper [same C. R. **246** (1958), 2574–2576; MR **20** #3971] by the same author in which he studies the problem of determining conditions under which a Taylor series admits the circle of convergence as a cut with unit probability. A number of theorems are proven of which the following is representative. Let $\{X_n\}$ be a sequence of essentially divergent independent random variables; then the series $\sum X_n z^n$ admits its circle of convergence as a cut with probability unity. In this theorem a sequence of random variables is said to be essentially divergent if there does not exist a sequence of numbers u_n such that $X_n - u_n \to 0$ with probability unity. The proof makes use of the idea of summability of a series of random variables by means of a real matrix $T=(a_{ij})$ for which $\lim_j a_{ij}=1$, $i=1, 2, \cdots$.

V. F. Cowling (Lexington, Ky.)

3544:

Tumarkin, G. C.; and Havinson, S. Ya. Properties of extremum functions in extremum problems for certain classes of analytic functions with a weighted metric. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 215–218. (Russian)

This paper announces a number of results most of which are treated in greater detail in the paper reviewed below. An exception to this statement is the use of a "complex" Green's function representing the extremal functions in terms of their moduli on the boundary.

A. J. Macintyre (Cincinnati, Ohio)

3545:

Tumarkin, G. C.; and Havinson, S. Ya. Investigation of the properties of extremal functions by means of duality relations in extremal problems for classes of analytic functions in multiply connected domains. Mat. Sb. N.S. **46 (88)** (1958), 195–228. (Russian)

The paper is concerned with the relation

$$(*)\qquad \sup\left|\int_\Gamma f(\zeta)\omega(\zeta)d\zeta\right| = \inf\int_\Gamma \rho(\zeta)|\omega(\zeta)-\varphi(\zeta)|d\zeta,$$

where Γ is the boundary of a finitely connected plane domain G, $\omega(\zeta)$ is a prescribed complex function and $\rho(\zeta)$ a prescribed real and non-negative function defined on Γ. The supremum is taken over all functions $f(\zeta)$ regular in G, having $\rho(\zeta)$ as an essential upper bound on Γ and moreover belonging to a special class $D(G)$ which becomes the class of functions of bounded type when G is a circle. The infimum is taken over functions $\varphi(\zeta)$ regular in G and also restricted to a certain very general class $E\{1/\rho(\zeta), 1, G\}$. If the supremum is taken for functions $f(\zeta)$ in this class $E\{1/\rho(\zeta), 1, G\}$, then (*) is replaced by

$$\sup\left|\int_\Gamma f(\zeta)\omega(\zeta)d\zeta\right| = \inf \operatorname{vrai\,max} \rho(\zeta)|\omega(\zeta)-\varphi(\zeta)|$$

(vrai max means essential upper bound for ζ on Γ, and $\varphi(\zeta)$ is restricted substantially as $f(\zeta)$ was previously).

Similar relations hold when $f(\zeta)$ is restricted to a class bounded in mean such as $\int_\Gamma |f(\zeta)/\rho(\zeta)|^p ds \leq 1$.

The length and complexity of the present paper (twenty-two theorems are listed, many with substantially independent converses) is due to the attempt to achieve generality for the region G and also the widest possible classes in which the functions $f(\zeta)$, $\varphi(\zeta)$, $\rho(\zeta)$, $\omega(\zeta)$ may be located. In some respects this aim is shown to be achieved.

The title gives an indication of the methods used. It is argued that the relation between a pair of extremal functions is a local property of a small arc of the boundary of G.

For the extremal problem under simpler conditions see F. F. Bonsall [J. London Math. Soc. **31** (1956), 105–110; MR **17**, 988] and Rogosinski and Shapiro [Acta Math. **90** (1953), 287–318; MR **15**, 516]. For the properties of classes of functions bounded in mean according to various definitions see Rudin [Trans. Amer. Math. Soc. **78** (1955), 46–66; MR **16**, 810] and Havinson [Mat. Sb. (N.S.) **36 (78)** (1955), 445–478; Dokl. Akad. Nauk SSSR **101** (1955), 421–424; MR **17**, 247, 248].

A. J. Macintyre (Cincinnati, Ohio)

3546:

Hvedelidze, B. V. The discontinuous Riemann-Privalov problem with given displacement. Soobšč. Akad. Nauk Gruzin. SSR **21** (1958), 385–389. (Russian)

The author studies the solubility of the functional equation

$$\phi^+[\alpha(t)] = a(t)\phi^-(t)+b(t),$$

where ϕ^+ is analytic outside and continuous on a closed curve Γ and ϕ^- is analytic inside and continuous on Γ. The conditions are the following: $\alpha(t)$ maps Γ on itself in a one-to-one way, preserving the arc-length, $\alpha'(t)\neq 0$ inside and satisfies a uniform Hölder condition. The function $a(t)=\chi^+[\alpha(t)]/\chi^-(t)$ where χ^+, $(\chi^+)^{-1}$ are analytic inside Γ and in $L_p(\Gamma)$ while χ^-, $(\chi^-)^{-1}$ are analytic outside Γ and in $L_q(\Gamma)$; $a\in L_1(\Gamma)$, $b/\chi^+\in L_p(\Gamma)$; $\phi^+, \phi^-\in L_p(\Gamma)$;

$$\phi^+(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\phi^+(t)}{z-t}dt+P(z),$$

and similarly for ϕ^- with $Q(z)$ replacing $P(z)$. The two functions P and Q represent the principal parts of a finite number of poles.

The author shows that the solution can be represented in the form

$$\phi^+(z) = \frac{\chi^+(z)}{2\pi i}\int_\Gamma \frac{\varphi(\beta(t))}{t-z}dt + \frac{\chi^+(z)}{2\pi i}\int_\Gamma \frac{f(\beta(t))}{\chi^+(t)(t-z)}dt+\chi^+(z)P(z),$$

where φ is a solution of the integral equation

$$\varphi(t)+\frac{1}{2\pi i}\int_\Gamma\left[\frac{\alpha'(t)}{\alpha(t)-\alpha(\tau)}-\frac{1}{\tau-t}\right]\varphi(\tau)dt = \frac{1}{2}f(t)-\frac{1}{2\pi i}\int\frac{f(\beta(t))}{\tau-t}dt+Q(t)+P(\alpha(t)).$$

Here β is the inverse function to α and $f(t)=b(t)/\chi^+(\alpha(t))$.

František Wolf (Berkeley, Calif.)

3547:

Gohberg, I. C.; and Krein, M. G. On the stability of a system of partial indices of the Hilbert problem for several unknown functions. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 854–857. (Russian)

The homogeneous Hilbert problem calls for vector-functions $\Phi_+(z)$, $\Phi_-(z)$, defined and analytic inside and outside a contour Γ, respectively, and satisfying on Γ the relation $\Phi_+(t)=A(t)\Phi_-(t)$, where A is a prescribed non-singular matrix-function, satisfying Hölder conditions. To obtain the "partial indices" $\kappa_1\geq\kappa_2\geq\cdots\geq\kappa_n$ one

chooses n non-trivial solutions $\Phi_-(z)=\chi_1(z), \cdots, \chi_n(z)$ such that $\chi_r(z)$ has the smallest possible order of magnitude $O(z^{-\kappa_r})$ at infinity, subject to $\chi_r(z)$ not being a linear combination, with polynomial coefficients, of $\chi_1(z), \cdots, \chi_{r-1}(z)$. According to G. F. Mandžavidze [Soobšč. Akad. Nauk Gruzin. SSR **14** (1953), 577–582; MR **15**, 785] the partial indices are stable under perturbation of $A(t)$; according to the present paper, this is true only if the partial indices have the form $\kappa_1=\cdots=\kappa_r=q+1$, $\kappa_{r+1}=\cdots=\kappa_n=q$. On the other hand, matrices $A(t)$ with this property are dense in the set of non-singular matrix-functions.

F. V. Atkinson (Canberra)

3548:

Saginyan, A. L. An inequality in the theory of analytic functions and some of its applications. I. Akad. Nauk Armyan. SSR. Dokl. **27** (1958), 257–262. (Russian. Armenian summary)

The following principal results are announced. Let $f(z)$ be analytic in $|z|<1$ and let $\int_0^{\phi}|\log|f(\rho e^{i\theta})||d\theta$ be uniformly absolutely continuous for ρ near 1. Let $\omega(t)$ be a continuous decreasing function on $0\leq t\leq 1$ such that $\Omega=\int_E(1-t)^{-1}\omega(t)dt<\infty$, $\omega(0)=1$, where E is a given measurable set. Then for $0<|z|<1$

$$|f(z)| < a\exp\left\{\frac{1-|z|}{4\Omega}\int_E \omega(|z|)\log|f(z)|d|z|\right\} \times\exp\left\{\frac{1}{\pi(1-|z|)}\int_0^{2\pi}\log^+|f(e^{i\phi})|d\phi\right\},$$

where $\int_E$ is extended over that part of an arc connecting the origin to $|z|=1$ on which $|z|\in E$ (the arc being cut only once by each circumference $|z|=\text{constant}$), and

$$a=\exp\left\{\frac{13+4\Omega}{\pi^{-1}\Omega}\left[1+(2\pi)^{-1}\int_0^{2\pi}\log^+|f(e^{i\phi})|d\phi\right]\right\}.$$

Hence $f(z)\equiv 0$ if $\int_E\omega(|z|)\log|f(z)|d|z|=-\infty$.

The author does not mention the possible connections between his results and many similar ones for half-planes.

R. P. Boas, Jr. (Evanston, Ill.)

3549:

Fuka, Jaroslav. Bemerkung zum Prinzip von Phragmén-Lindelöf. Časopis Pěst. Mat. **84** (1959), 64–73. (Czech. Russian and German summaries)

The author uses conformal representations to prove results of Phragmén-Lindelöf type for sub-harmonic functions, generalising the known cases in which the exceptional point may be an angular point or a cusp formed by circular arcs.

F. V. Atkinson (Canberra)

3550:

Gaier, Dieter. Über ein Extremalproblem der konformen Abbildung. Math. Z. **71** (1959), 83–88.

The author considers doubly-connected plane regions of conformal modulus M and defines $D(G)$ to be the diameter of the inner boundary of G, while $A(G)$ stands for the area enclosed by the outer boundary of G. It is established that

$$D^2/A \leq (16/\pi)\sum M^{2n}/[n(M^{4n}-1)],$$

where the summation is extended over all positive odd integers n, and the extremal regions G for which equality holds are exhibited explicitly.

P. R. Garabedian (New York, N.Y.)

3551:

Kubo, Tadao. Hyperbolic transfinite diameter and some theorems on analytic functions in an annulus. J. Math. Soc. Japan **10** (1958), 348–364.

Let a, b be two points in $|z|<1$ and write

$$[a,b]=\left|\frac{a-b}{1-\bar{a}b}\right|.$$

For any compact sub-set E of $|z|<1$ the hyperbolic transfinite diameter $\tau(E)$ was defined by Tsuji [Jap. J. Math. **19** (1947), 483–516; Comment. Math. Univ. St. Paul. **4** (1955), 1–3; MR **10**, 365; **17**, 473], as follows: Set

$$d_n=d_n(E)=\left\{\max_{p_i\in E,\,p_j\in E}\prod_{1\leq i<j\leq n}[p_ip_j]\right\}^{1/\binom{n}{2}},$$

where $p_1, p_2, \cdots, p_n$ are points in E. Then the sequence d_n is decreasing and

$$\tau(E)=\lim_{n\to\infty}d_n(E).$$

Tsuji also proved that the hyperbolic transfinite diameter is equal to the capacity of the condenser characterised by the sets E and $|z|=1$. Using these facts the author proves the following theorem.

Let $\mathscr{F}$ be the class of functions $w=f(z)$ which are regular in D: $Q<|z|\leq 1$; whose range of values, D_f, for z in D, lies in $|w|<1$; and which further satisfy $|f(z)|=1$ for $|z|=1$. Let E_f be the complement of D_f in $|w|<1$. Then

$$(*)\qquad \tau(E_f)\leq Q.$$

Equality holds if and only if $f(z)$ is univalent in D.

This theorem, together with the geometrical definition of $\tau(E)$, enables the author to prove a variety of theorems for his functions regular in an annulus which are largely generalisations of corresponding results obtained previously by the reviewer for functions regular inside or outside a circle [see, e.g., J. Analyse Math. **1** (1951), 155–179; MR **13**, 545]. Suppose, for instance, that $M<Q$, that $w=f_0(z)$ maps the annulus $Q<|z|<1$ (1, 1) conformally onto the annulus $M<|w|<1$ cut from $-r_0$ to $-M$ along the real axis. Then if $f\in\mathscr{F}$, f is not equivalent to $f_0(z)$ by rotations in the z and w plane, and $\left|\frac{f}{w}\right|>M$, then $w=f(z)$ assumes all values w on some circle $|w|=r$ with $r<r_0$.

W. K. Hayman (London)

3552:

Kubo, Tadao. On a theorem of analytic functions in an annulus. Math. Japon. **5** (1958/59), 17–20.

With the notation of the paper reviewed above, let

$$\zeta=\xi+i\eta=l(w)=\frac{1}{\pi}\log\frac{1+w}{1-w}+\frac{i}{2}$$

map $|w|<1$ onto the strip $0<\eta<1$ in the ζ-plane, and consider the class of functions $\mathscr{F}'$ of the form $\zeta=\phi(z)=l[f(z)]$ where $f(z)\in\mathscr{F}$. Let E be a connected set of values in the ζ-plane not assumed by $\zeta=\phi(z)\in\mathscr{F}$ and lying in the strip $0<\eta<1$. By using the inequality (*) the author obtains the exact upper bound for the oscillation of $\xi=\mathscr{R}\zeta$ on E. The bound is attained when $\phi(z)$ maps the annulus $Q<|z|<1$ onto the strip $0<\eta<1$ cut along a segment of the line $\eta=\frac{1}{2}$.

W. K. Hayman (London)

3553:

Landau, H. J.; and Osserman, R. Some distortion theorems for multivalent mappings. Proc. Amer. Math. Soc. **10** (1959), 87–91.

Elementary proofs, based on the Cauchy-Riemann equations and the maximum principle, are given for some distortion theorems. One main result is: Let W_i, $i=1, 2$, be two arbitrary Riemann surfaces. Let R_i be a relatively compact region on W_i, bounded by a finite number of Jordan curves, and let C_i be a distinguished boundary curve of R_i. Let u_i be the harmonic measure of C_i with respect to R_i, and let p_i be the period around C_i of the harmonic conjugate to u_i. If $f(p)$ is an analytic map of R_1 into R_2 and if for every curve C_1^* homologous to C_1 the image curve $f(C_1^*)$ is homologous to nC_2 for some integer n, then $p_1 \geqq |n| p_2$. Extensions and applications of this result are briefly indicated.

M. Schiffer (Stanford, Calif.)

3554:

Kakehashi, Tetsujiro. On schlicht functions. I. Proc. Japan Acad. **35** (1959), 134–136.

Let $\{\mu_m\}_1^\infty$ and $\{\lambda_m\}_0^\infty$ be sequences of positive and of real constants, respectively. Let

$$F_0(z) = [(1+z)/(1-z)+i\lambda_0]^2,$$

and for $k=1, 2, \cdots$, let

$$F_k(z) = \{[F_{k-1}(z)+\mu_k]^{1/2}+i\lambda_k\}^2 = \sum_{n=0}^{\infty} A_n^{(k)} z^n.$$

Then $F_k(z)$ maps the unit disk univalently onto a domain which is obtained by removing from the w-plane a slit consisting of a half-line with a finite number of finite branches. For $k=1, 2, \cdots$, the author establishes the inequality $|A_1^{(k)}|>4$ and the asymptotic relation $A_n^{(k)} \sim cnA_1^{(k)}$; here c depends on the λ_m and the μ_m, but satisfies the condition $|c|<1$. *G. Piranian* (Ann Arbor, Mich.)

3555:

Myrberg, P. J. Reduktion der Verzweigungspunkte Riemannscher Flächen durch konforme Abbildung. Ann. Acad. Sci. Fenn. Ser. A. I **261** (1959), 7 pp.

The author considers the problem of representing an open Riemann surface as a covering surface of the plane with smallest possible orders of the algebraic branch points.

Let Γ be a totally discontinuous group of linear transformations with the fundamental region B, and let $x(z)$ be a meromorphic function in B, automorphic under Γ. Using Poincaré series, the author constructs another automorphic function $y(z)$ such that $x(z)$ and $y(z)$ take on every pair of values at most once, except for a countable set. The following complete result is then established: Every open Riemann surface can be conformally mapped onto a covering surface R of the extended plane such that all algebraic branch points of R are of order one.

L. Sario (Los Angeles, Calif.)

3556:

Sario, Leo. Countability axiom in the theory of Riemann surfaces. Ann. Acad. Sci. Fenn. Ser. A. I, no. 250/32 (1958), 7 pp.

This paper gives a proof of the Radó countable basis theorem for Riemann surfaces which is based on the observation that it suffices to establish the existence of a non-constant harmonic function on a Riemann surface less a disk in order to establish the Radó theorem. This observation is a consequence of the fact that a 2-dimensional manifold admitting an interior transformation into a surface is itself a surface. The existence of a non-constant harmonic function is established by the author's operator method [Trans. Amer. Math. Soc. **73** (1952), 459–470; MR **14**, 863]. It is also shown that the author's reduction theorem [ibid. **79** (1955), 362–377; MR **19**, 846] employed in the study of extremal problems may be established by use of directed limits without appeal to exhaustion methods. *M. H. Heins* (Urbana, Ill.)

3557:

Osserman, Robert. Koebe's general uniformization theorem: The parabolic case. Ann. Acad. Sci. Fenn. Ser. A. I **258** (1958), 7 pp.

The distinction between parabolic and hyperbolic types commonly used in conformal mappings of simply connected Riemann surfaces has not been employed in the proof of Koebe's general uniformization theorem. E.g., the principal function p_0 has a priori a single-valued conjugate p_0^* [L. Sario, Trans. Amer. Math. Soc. **72** (1952), 281–295; MR **13**, 735], and the function $p_0+ip_0^*$ gives the various canonical mappings in all cases.

The author discusses a possibility of carrying the type dichotomy over to Koebe's theorem. He presupposes the existence of a harmonic function u bounded near the ideal boundary and possessing a suitable singularity. He then shows, making use of a well-known lemma, that u^* is single-valued and $u+iu^*$ univalent if the surface is parabolic. *L. Sario* (Los Angeles, Calif.)

3558:

Rahman, Q. I. A property of entire functions of small order and finite type. Duke Math. J. **26** (1959), 407–408.

"The object of this note is to prove the following theorem: Let $f(z)$ be an entire function of order ρ $(0<\rho<2)$ and finite type. If $f(0)=1$, then

$$\int_R^S \{(\log|f(z)|)/|z|^{1+\rho}\}d|z|$$

is bounded as R and S tend to infinity with $R<S$ and $S/R=O(1)$ and z varies along any path joining R and S." (Author's summary) *A. G. Azpeitia* (Amherst, Mass.)

3559:

Ibragimov, I. I. Extremal problems in a class of entire functions of finite degree. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 243–256. (Russian)

The author extends his previous work [Uspehi Mat. Nauk (N.S.) **12** (1957), no 3 (75), 323–328; MR **19**, 737] on inequalities for entire functions of exponential type from the case when the function belongs to L^2 on the real axis to the case when it belongs to L^p, mostly for $1 \leqq p \leqq 2$. The class $W^{(p)}$ consists of entire functions of exponential type ν with finite L^p norm on the real axis; if $1 \leqq p \leqq 2$, they can then be represented in the form $f(z)=(2\pi)^{-1/2}\int_{-\nu}^{\nu} e^{izu}\phi(u)du$. The author considers functionals of the form $J(f)=(2\pi)^{-1/2}\int_{-\nu}^{\nu} \phi(u)\Phi(u)du$, with auxiliary conditions on Φ that do not appear to be essential. He proves that $|J(f)| \leqq (2\pi)^{-1/p}\|f\|_p\|\Phi\|_{p'}$, and applies this to many special cases, obtaining among others bounds for f, for $\int_\alpha^\beta f(x)dx$, for $|f(x+\delta) \pm f(x-\delta)|$, and for $\bar{f}(x)$. He proves that if

$0<p\leq 2$ and $p'\geq p$ then $\|f\|_{p'}\leq (\nu/\pi)^{1/p-1/p'}\|f\|_p$, and states that when $p>2$ it is true that $\|f\|_{p'}\leq (\nu p/\pi)^{1/p-1/p'}\|f\|_p$.
R. P. Boas, Jr. (Evanston, Ill.)

3560:

Singh, Shri Krishna. Sur la densité du nombre des points où une fonction entière prend une valeur donnée. Bull. Sci. Math. (2) **82** (1958), 12–14.

Let $f(z)$ be an entire function of finite positive order ρ. If a and b are two different finite numbers, the author shows that

$$\limsup \frac{N(r,a)+N(r,b)}{\log M(r)} \geq \frac{1-\rho+(1+\rho^2)^{1/2}}{1+\rho+(1+\rho^2)^{1/2}}\left\{\frac{\rho}{1+(1+\rho^2)^{1/2}}\right\}^{\rho} e^{-1}.$$

Hence $\limsup N(r,a)/\log M(r)>0$ except for at most one value of a. *R. P. Boas, Jr.* (Evanston, Ill.)

3561:

Künzi, H. P.; and Wittich, H. The distribution of the a-points of certain meromorphic functions. Michigan Math. J. **6** (1959), 105–121.

Let W be a simply connected open Riemann surface, spread over the w-sphere. Two special classes of W are formed of those surfaces whose graphs (in the sense of Speiser, Nevanlinna and Elfving) are formed of a finite number of simply periodic ends or doubly periodic ends. The authors obtain some geometric properties of the distribution of a-points of a meromorphic function $w=f(z)$ in one of the two classes described. *M. Tsuji* (Tokyo)

3562a:

Hiong, King-lai. Sur un problème de M. Montel concernant la théorie des familles normales de fonctions. Sci. Record (N.S.) **2** (1958), 189–192.

3562b:

Hiong, King-lai. Deux théorèmes sur les fonctions holomorphes dans un cercle avec ces valeurs exceptionnelles. Sci. Record (N.S.) **2** (1958), 239–243.

3562c:

Hiong, King-lai. Quelques théorèmes sur les fonctions méromorphes admettant un ensemble de valeurs déficientes. Sci. Record (N.S.) **3** (1959), 61–64.

In these three papers the author seeks to find bounds of as simple a kind as possible for the growth of a function, when there are restrictions on the number of times the function and its derivatives assume certain values. In the first paper the author proves quite shortly a more precise version of a theorem of Miranda [Bull. Soc. Math. France **6** (1935), 185–196].

Theorem: Suppose that $f(z)$ is regular in the unit circle and that $f(z)\neq 0$, $f^{(k)}(z)\neq 1$ there. Then we have

$$\log M(r,f) < \frac{1}{1-r}\left[H_k\log^+|f(0)|+H_k\log^+\left|\frac{1}{f(0)}\right|+K_k\log\frac{2}{1-r}\right],$$

where H_k and K_k depend only on k.

In the next paper the hypothesis $f^{(k)}(z)\neq 1$ is replaced by

$$N\left(r,\frac{1}{f^{(k)}-1}\right)\leq \lambda\log\frac{1}{1-r},\quad 0<r<1,$$

where $k=0$ or 1. If $f(z)=c_0+c_1z+\cdots$, the author obtains bounds for c_{k+1} in terms of λ, c_0 and c_k only.

In the third paper hypotheses on the defects of $f(z)$ and its derivative result in conclusions which are similar to the above but more complicated.

It may be worth pointing out that by using methods of the reviewer [Acta Math. **86** (1951), 89–257; MR **13**, 546; Theorem II, p. 129] the last two terms in the right hand side of the above theorem could probably be replaced by K_k (as is the case when $k=0$). A similar remark applies to the second paper. *W. K. Hayman* (London)

3563:

Fan, Wui-kwok. Sur quelques classes de fonctions méromorphes quasi exceptionnelles. Sci. Record (N.S.) **3** (1959), 1–5.

Eine meromorphe Funktion $f(z)$ heisst "quasi-exzeptionell" ["quasi-exzeptionell von endlicher Totalordnung"], wenn die Funktionenfamilie $f_n(z)=f(2^nz)$ quasi-normal [quasi-normal von endlicher Totalordnung] ist im Kreisring Γ: $\frac{1}{2}\leq|z|\leq 2$. C_{p+p+p} sei die Bedingung: $f(z)$ nimmt die Werte 0, 1, ∞ höchstens p mal an in jedem Kreisring Γ_n: $2^{n-1}\leq|z|\leq 2^{n+1}$. Eine meromorphe Funktion, die C_{p+p+p} erfüllt, ist quasi-normal von endlicher Totalordnung, sie ist Quotient zweier ganzer Funktionen der Ordnung Null.

Es seien a, b, c beliebige verschiedene komplexe Zahlen; a_μ, b_μ, c_μ seien geeignete Folgen von verschiedenen a-, b-, c-Stellen der Funktion $f(z)$. Notwendig dafür, dass $f(z)$ quasi-exzeptionell sei, ist: wenigstens einer der Quotienten a_μ/b_μ, a_μ/c_μ, b_μ/c_μ hat den Limes Eins. Nach einer Methode von P. Montel [*Leçons sur les familles normales*, Gauthier-Villars, Paris, 1927, p. 168ff.] wird gezeigt, dass

$$f(z)=z^m\prod_\lambda(1-z/a_\lambda)\Big/\prod_\mu(1-z/b_\mu)$$

quasi-exzeptionell von endlicher Totalordnung ist, wenn folgende vier Bedingungen erfüllt sind: (I) Die Differenz zwischen der Anzahl der Pole und den Nullstellen innerhalb jedes Kreises $|z|\leq r$ ist beschränkt; (II) die Anzahl der Pole und der Nullstellen in Γ_n ist beschränkt; (III) die Zahlen

$$|a_p|^m\prod_{|a_\lambda|<|a_p|}|a_p/a_\lambda|\Big/\prod_{|b_\mu|<|a_p|}|a_p/b_\mu|,$$

$$|b_q|^{-m}\prod_{|b_\mu|<|b_q|}|b_q/b_\mu|\Big/\prod_{|a_\lambda|<|b_q|}|b_q/a_\lambda|$$

sind beschränkt für beliebiges p und q; (IV) es gibt eine Folge a_{λ_ν} von Nullstellen und eine Folge b_{μ_ν} von Polen, sodass $\lim_\nu (a_{\lambda_\nu}/b_{\mu_\nu})=1$. *F. Huckemann* (Giessen)

3564:

Kulshrestha, P. K. On growth of derivatives of admissible functions. Proc. Nat. Inst. Sci. India. Part A **25** (1959), 75–76.

Let $f(z)=\sum_0^\infty a_nz^n$ be admissible in the sense of the reviewer [J. Reine Angew. Math. **196** (1956), 67–95; MR

18, 293] in $|z|<R$, where $0<R\leqq\infty$, and set $\alpha(r)=rf'(r)/f(r)$. Then the reviewer showed that

$$r^j f^{(j)}(r) \sim [\alpha(r)]^j f(r) \quad (r \to R-).$$

The present author deduces from this in effect that

$$r^j f^{(j)}(r)-[\alpha(r)]^j f(r) = O[\alpha(r)]^{j+\frac{1}{2}+\epsilon} f(r),$$

as $r\to R$ through a sequence of values. The fact that $\alpha(r)\to\infty$ would have helped in this deduction. In spite of kind acknowledgements to the reviewer, the author's achievement is entirely his own.

W. K. Hayman (London)

3565:

Flett, T. M. On the radial order of a univalent function. J. Math. Soc. Japan **11** (1959), 1–3.

By means of the Cesàro mean of order $\alpha>1/2$, the author gives an elementary proof of the following Denjoy-Seidel-Walsh theorem: Let $f(z)$ be regular and univalent in $|z|<1$, then for almost all θ, $f'(z)=o((1-|z|)^{-1/2})$ uniformly as $z\to e^{i\theta}$ in each Stolz domain.

M. Tsuji (Tokyo)

3566:

Scott, W. T. Comment on a paper of C. Uluçay. Proc. Amer. Math. Soc. **10** (1959), 395.

The author reviews the paper by Uluçay [Proc. Amer. Math. Soc. 8 (1957), 923–925] on the Bloch-Landau constant of the third kind, commenting in detail on a point mentioned in the review [MR **19**, 736].

E. Reich (Minneapolis, Minn.)

3567:

Zmorovič, V. A. On the theory of special classes of univalent functions. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 5–9. (Ukrainian. Russian and English summaries)

The author gives representation formulas for some classes of univalent functions considered by Robertson [Ann. of Math. (2) **38** (1937), 770–783] and Rahmanov [Dokl. Akad. Nauk SSSR **78** (1951), 209–211; MR **12**, 816], and for generalizations of classes considered by Čakalov [C. R. Acad. Sci. Paris **242** (1956), 437–439; MR **17**, 724] and Thale [Proc. Amer. Math. Soc. **7** (1956), 232–244; MR **17**, 1063]. (1) Let $F(z, \theta)$ be continuous in both arguments for $|z|\leqq 1$ and $0\leqq\theta\leqq 2\pi$. Let F be analytic for $|z|<1$ and $\Re F(z, \theta)>0$ for $|z|<r\leqq 1$. Then if $\phi(z)$ is univalent and maps $|z|<1$ on a convex set, $\mu(\theta)$ is of bounded variation on $[0, 2\pi]$, and $\Phi(z, \theta)=\int_0^z \phi'(\zeta)F(\zeta, \theta)d\zeta$, the function $f(z)=\int_0^{2\pi}\Phi(z, \theta)d\mu(\theta)$ is analytic in $|z|<1$ and univalent in $|z|<r$, but not necessarily in a larger disk. (2) A region is of class $B(\alpha;\beta)$ $(0\leqq\alpha\leqq\pi;\ -\pi\leqq\beta\leqq 0)$ if with $x+iy$ it contains all points $x-(y-t)\cot\alpha+it$ $(0\leqq t\leqq y)$ if $y>0$; $x-(y-t)\cot\beta+it$ $(y\leqq t\leqq 0)$ if $y<0$; when $\beta=-\alpha$ and the region is symmetric about the real axis, we have $B^*(\alpha;-\alpha)$. A function is of class $B(\alpha;\beta)$ if it is analytic and univalent in $|z|<1$ and maps this disk on a region of class $B(\alpha;\beta)$. A necessary and sufficient condition that $f\in B(\alpha;\beta)$ is

$$e^{i\sigma}(1-ze^{-i\gamma})^{1-\lambda}(1-ze^{-i\delta})^{1+\lambda}f'(z) = ih+c\int_0^{2\pi}\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\,d\mu(\theta),$$

where $\mu(\theta)$ has total variation at most 1; $\lambda=(\alpha-\beta)/\pi$; γ, δ, σ, h, c are real; $c>0$; $\gamma<\delta<\gamma+2\pi$; $\sigma=\frac{1}{2}\gamma(1-\lambda)+\frac{1}{2}\delta(1+\lambda)+\pi\lambda-\alpha$. (3) If $f(z)\in B^*(\pi/2;-\pi/2)$, $f(0)=0$, $\mu(t)$ is of bounded variation, then

$$\int_{-1}^{1} t^{-1}f(tz)d\mu(t)\in B^*(\pi/2;\ -\pi/2).$$

(4) If $\phi(z)$ is univalent and maps $|z|<1$ on a convex set, $\alpha+\beta=1$, $\beta/\alpha=\lambda e^{i\theta}$, $\lambda>0$, $|\theta|<\pi/2$, then $\alpha\phi(z)+\beta z\phi'(z)$ belongs to the class for which $\Re[e^{i\tau}f'(z)/\psi'(z)]>0$ with τ real and ψ univalent and convex.

R. P. Boas, Jr. (Evanston, Ill.)

3568:

Kas'yanyuk, S. A. On the method of structural formulae and the principle of conformity of boundaries in conformal mapping. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 14–17. (Ukrainian. Russian and English summaries)

If $a(u, v)$ is harmonic in a region G, the functions $f(z)$ with $f(0)=0$, $f'(0)=1$, mapping $|z|<1$ on G so that $\Re\{zf'(z)[a_v'+ia_u']\}\geqq 0$, are characterized by the formula

$$f(z) = \int_0^z\left\{\frac{\Re\psi(0)}{2\pi\psi(z)}\int_{-\pi}^{\pi}\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\,d\mu(\theta)\right\}dz+i\Im\psi(0)\int_0^z\frac{dz}{\psi(z)},$$

where μ is of bounded variation, $a_v'+ia_u'=z^{-1}\psi(z)$. Several examples are given.

R. P. Boas, Jr. (Evanston, Ill.)

3569:

Zmorovič, V. A. On bounds for the variation of the curvature of the image of a plane curve under a univalent conformal mapping. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 351–354. (Ukrainian. Russian and English summaries)

Let ζ be a fixed point in a region D of the complex z plane, let L be a curve through ζ with curvature K, and unit tangent vector τ at ζ. If $f(z)$ is regular and univalent in D taking the curve L into L^* and the point ζ into ω, then the curvature of L^* at ω is

$$K^* = \frac{1}{|f'(\zeta)|}\left\{K+\Im\left(\tau\frac{f''(\zeta)}{f'(\zeta)}\right)\right\}.$$

From this formula the author deduces some general theorems giving sharp bounds for K^* in terms of bounds for $|f'(\zeta)|$ and $|f''(\zeta)/f'(\zeta)^2|$ for functions regular and univalent in D. He obtains a sharp lower bound when D is the unit circle and $f(z)$ is suitably normalized.

A. W. Goodman (Lexington, Ky.)

3570:

Brown, Richard K. Typically-real functions. Canad. J. Math. **11** (1959), 122–130.

$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ $(z=x+iy)$ is called typically real of order one in $|z|\leqq R$ if (i) $f(z)$ is regular in $|z|\leqq R$, (ii) $\mathscr{I}(f(z))>0$ if and only if $y>0$; we denote this by $f(z)\in T_1(R)$. $f(z)$ is called typically real of order p (>1) in $|z|\leqq R$ if (i) $f(z)$ is regular in $|z|\leqq R$, (ii) there exists ρ $(0<\rho<R)$ such that, on any circle $|z|=r$ $(\rho<r<R)$, $\mathscr{I}(f(z))$ changes its sign exactly $2p$ times; we denote this by $f(z)\in T_p(R)$. In the main theorem, the author proves a recursion formula for R_p $(p=1, 2, \cdots)$ such that $f(z)\in T_p(1)\Rightarrow f(z)\in T_1(R_p)$.

M. Tsuji (Tokyo)

3571:

Chang, Kai-ming. Functions typically real and meromorphic in a circular ring. Acta Math. Sinica **9** (1959), 37–50. (Chinese. English summary)

In 1932 W. Rogosinski [Math. Z. **35** (1932), 93–121] studied the class of functions typically-real in the unit circle. In 1956 A. W. Goodman [Trans. Amer. Math. Soc. **81** (1956), 92–105; MR **17**, 724] studied functions typically-real and meromorphic in the unit circle. This paper is an extension of Goodman's result to the case of a circular ring R_q: $q<|z|<1$. The class of functions in R_q is denoted by $T_r(R_q)$. The author defines

$$S_q(z, a) = \sum_{\nu=-\infty}^{\infty} (q^{2\nu}z)/(1-2aq^{2\nu}z+q^{4\nu}z^2) \quad (a \text{ real}),$$

which is schlicht and typically-real in R_q with absolute value less than $(q+q^{-1})/2$. For $|a|<(q+q^{-1})/2$, the function $w=S_q(z, a)$ maps the upper half ring of R_q onto the upper half plane of w. Its inverse function is expressed in terms of elliptic integral. Two certain classes of curves $\{l_a\}$ and $\{l_b\}$ in the upper half ring of R_q are introduced. Then the modulus of any function belonging to $T_r(R_q)$ on l_b is used to estimate that in R_q. Most of Goodman's results are extended to this case. In some parts Goodman's results are carried over almost word for word.

C. Y. Wang (Minneapolis, Minn.)

3572:

Makmak, K. M. On a property of functions with limited boundary rotation. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 567–570. (Ukrainian. Russian and English summaries)

In this note the author establishes the boundaries of the change in curvature of the level lines on mapping a unit circle by functions with limited boundary rotation. The estimates found are exact. *Author's summary*

3573:

Mikhail, M. N. On linear differential basic sets of polynomials. Proc. Math. Phys. Soc. Egypt. no. 21 (1957), 19–23 (1958). (Arabic summary)

Let $\{p_n(z)\}$ be a basic set of polynomials of Whittaker order ω. The author studies the order and effectiveness of the polynomial set $\{q_n\}$, where

$$q_n(z) = \left\{1+\frac{d}{dz}+\left(\frac{d}{dz}\right)^2+\cdots+\left(\frac{d}{dz}\right)^r\right\} p_n(z).$$

R. C. Buck (Princeton, N.J.)

3574:

Zaĭcev, M. N. Complete systems of entire analytic functions. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 4, 3–15. (Russian)

The completeness of systems $\{f(\alpha_k z)\}$, where f is an analytic function of some class, has been investigated by several authors. However, when the system is complete it is not usually true that a given analytic function can be represented by a convergent series of functions of the system. The author investigates cases in which a given function can be represented by a series of linear combinations of the functions of the system. First he considers the special case when $f(z)=\sum a_n z^n$ and $1/a_n=\int_0^\infty x^n d\tau(x)$, $\tau(x)=-\exp(-\sigma x^\rho)$, $0<\sigma<\infty$, so that $a_n=\sigma^{n/\rho}/\Gamma(n/\rho+1)$. Let $|\alpha_n|=\beta_n$ and $\bar\beta_n=\max_{0\le i\le n}\beta_i$. Then if $\bar\beta_n{}^\rho/n\to 0$, a representation of the desired form holds in any disk $|z|<R$; the remainder tends exponentially to 0 and vanishes (for large n) for a finite linear combination of $f(\alpha_n z)$. A similar result holds whenever $f(z)=\sum b_n z^n$ is entire of positive finite order ρ and positive finite type σ, with $b_n\neq 0$ and $n^{1/\rho}|b_n|^{1/n}\to(\sigma e\rho)^{1/\rho}$. Finally, for this more general case, $\{f(\alpha_n z)\}$ is complete if only $n/|\alpha_n|^\rho\to\infty$.

R. P. Boas, Jr. (Evanston, Ill).

3575:

Avetisyan, G. M. Approximation of analytic functions by entire functions. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **11** (1958), no. 6, 3–14. (Russian. Armenian summary)

M. Keldyš [Dokl. Akad. Nauk SSSR **47** (1945), 239–241; MR **7**, 150] showed that a function analytic in a rectilinear strip can be approximated uniformly by entire functions of a specified rate of growth. The author extends this work to curvilinear strips. Let D be a region containing the real axis, bounded by curves $y=\pm\phi(x)$, where $\phi(x)\downarrow 0$ as $|x|\uparrow\infty$; let D_R be the intersection of D with the disk $|z|\le R$. Let $f(z)$ be analytic in D and let $M(R)$ be the maximum modulus of f in D_R. If $\varepsilon>0$ there is an entire function $F(z)$ such that $|F(x)-f(x)|<\varepsilon$ for $-\infty<x<\infty$; and for $k\ge 2$ and $|z|\le R$,

$$|F(z)|<\left\{\frac{CRM(kR)}{\phi(kR)}\right\}^{1+C_1\phi(0)} \varepsilon^{-C_1\phi(0)}.$$

R. P. Boas, Jr. (Evanston, Ill.)

3576:

Dolženko, E. P. Construction on a nowhere dense continuum of a nowhere differentiable function which can be expanded into a series of rational functions. Dokl. Akad. Nauk SSSR **125** (1959), 970–973. (Russian)

If K is any nowhere dense continuum there is a function f that is representable on K as a uniform limit of rational functions and is nowhere differentiable on K.

R. P. Boas, Jr. (Evanston, Ill.)

3577:

Al'per, S. Ya. Asymptotic values of best approximation of analytic functions in a complex domain. Uspehi Mat. Nauk **14** (1959), no. 1 (85), 131–134. (Russian)

Examples concerning the problem of approximation of analytic functions in $|z|\le R$ by polynomials. The degree of approximation of $(z-a)^{-k}$, $k=1, 2, \cdots$, $|a|>R$ is asymptotically $\sim n^{k-1}R^{n+1}[(k-1)!(|a|^2-R^2)]^{-1}|a|^{-n-k-1}$ as $n\to\infty$; the degree of approximation of $z^s(z^p-a^p)^{-1}$, $p=1, 2, \cdots$, $s=0, \cdots, p-1$, $|a|>R$ is exactly given.

G. G. Lorentz (Syracuse, N.Y.)

3578:

Kočaryan, G. S. Approximation by rational functions in the complex plane. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **11** (1958), no. 4, 53–77. (Russian. Armenian summary)

L'auteur considère les problèmes suivants. 1. Étude des sommes partielles des séries de Fourier-Džrbašyan $\sum_{-\infty}^{\infty} C_k\phi_k(z)$, où $\{\phi_k\}$ est le système orthogonal sur $|z|=1$ associé par Džrbašyan à deux suites complexes $\{\alpha_k\}$ ($|\alpha_k|<1$, $k=1, 2, \cdots$) et $\{\beta_k\}$ ($|\beta_k|>1$, $k=-1, -2, \cdots$) [Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn.

Nauki 9 (1956), no. 7, 3–28; MR 18, 393]; approximation d'une fonction donnée sur $|z|=1$ par ces sommes partielles. 2. Approximation d'une fonction f analytique dans un ouvert et continue à la frontière, moyennant des conditions de régularité convenables, par des fractions rationnelles ayant des pôles imposés. 3. Rapidité d'approximation d'une telle fonction f par les sommes partielles des séries de fractions rationnelles introduites par Džrbašyan [ibid. 10 (1957), no 1, 21–29; MR 20 #3297]. 4. Propriétés d'analyticité de limites uniformes de fractions rationnelles ayant leurs pôles dans une suite donnée.

J. P. Kahane (Montpellier)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

See also 3466, 3884, 3885.

3579:

Leja, F. Sur le domaine de convergence des séries de polynômes homogènes de deux variables complexes. Ann. Polon. Math. 6 (1959/60), 93–98.

Let $A=(a, b)$ and $A'=(a', b')$ be two points in the space C^2 of two complex variables, not collinear with the origin. Define a "distance" in C^2 by $(A, A')=\frac{1}{2}|ab'-a'b|$, and for an arbitrary point $P=(x, y)$ in C^2, let

$$t(P)=\frac{(A, P)+(A', P)+[\{(A, P)+(A', P)\}^2-(B, P)^2]^{1/2}}{(A, A')}$$

where $B=A'-A$. Let J be the line segment joining A and A'. The author shows that if $\sum P_n(x, y)$ is a series of polynomials P_n homogeneous of degree n, which is bounded on J, then it converges uniformly on the sets defined by $t(P)\leq\rho<1$. The author also considers the case in which boundedness is given on two disjoint intervals J_1 and J_2 of the same line in C^2.

R. C. Buck (Princeton, N.J.)

3580:

Ivanov, V. K. A characterization of the growth of an entire function of two variables and its application to the summation of double power series. Mat. Sb. (N.S.) 47 (89) (1959), 3–16. (Russian)

In an earlier paper [Mat. Sb. 43 (85) (1957), 367–378; MR 20 #989] the author gave an n-variable extension of Pólya's relationship between an entire function of exponential type and the singular points of its Borel transform; this contained an auxiliary hypothesis that kept it from reducing to the original theorem for $n=1$. Here he gives an apparently stronger theorem for the two-variable case, with no additional hypothesis, and shows that it is actually equivalent to the earlier one. Let $F(z, w)=\sum\sum a_{mn}z^m w^n/(m!n!)$ be an entire function such that $|F(z, w)|\leq Ae^{\sigma|z|+\tau|w|}$. Let $f(z, w)=\sum\sum a_{mn}z^{-m-1}w^{-n-1}$. Let $z=re^{i\theta}$ and $w=se^{i\phi}$ and consider the two-dimensional planes $\Pi(\theta, \phi)$ in the 4-space of (z, w) obtained by specifying θ and ϕ. Let $T(\theta, \phi)$ be the set of points (μ, ν) in $\Pi(\theta, \phi)$ such that for each point (μ, ν) in $T(\theta, \phi)$ there is a constant $A=A(\mu, \nu)$ such that $|F(re^{i\theta}, se^{i\phi})|\leq Ae^{\mu r+\nu s}$ for all positive r and s. Let $C(\theta, \phi)$ be the set of points (c_1, c_2) of $\Pi(\theta, \phi)$ such that all points (z, w) for which $\Re(ze^{-i\theta})\geq c_1$ and $\Re(we^{-i\phi})\geq c_2$ are points of regularity of f. Then the closure of $T(\theta, \phi)$ coincides with the closure of $C(-\theta, -\phi)$.

Let $C_4(\theta, \phi)$ be the set (in 4-space) of points (z, w) such that the point $b=(b_1, b_2)$, where $b_1=\Re(ze^{-i\theta})$, $b_2=\Re(we^{-i\phi})$, belongs to $C(\theta, \phi)$. Then $f(z, w)=\iint_\Lambda e^{-(z\zeta+w\eta)}f(\zeta, \eta)d\zeta d\eta$, with Λ the two-dimensional ray $\zeta=\rho e^{i\alpha}$, $\eta=\sigma e^{i\beta}$, provided that (z, w) is an interior point of $C_4(-\alpha, -\beta)$; the integral diverges if (z, w) is an exterior point of this region.

Application to two-dimensional Borel summability: Let $g(w, z)=\sum\sum a_{mn}z^n w^m$ converge in a neighborhood of the origin and associate with it the entire function

$$G(w, z)=\sum_{l=0}^{\infty}\frac{1}{(l+1)!}\sum_{m+n=l}a_{mn}w^m z^n.$$

The Borel sum of the original series is defined to be $\int_0^\infty e^{-t}tG(tw, tz)dt$. The author obtains the following analogue of the Borel polygon (which is not necessarily convex). Let (a, b) be a singular point of $g(w, z)$. Let $D(a, b)$ be the set of points (w, z) for which at least one of the conditions $\Re(w/a)\leq 1$ or $\Re(z/b)\leq 1$ is satisfied. The intersection of the regions $D(a, b)$ is the region of Borel summability of $\sum\sum a_{mn}z^n w^m$.

R. P. Boas, Jr. (Evanston, Ill.)

3581:

Pyateckiĭ-Šapiro, I. I. Discrete subgroups of the group of analytic automorphisms of the polycylinder, and automorphic forms. Dokl. Akad. Nauk SSSR 124 (1959), 760–763. (Russian)

Let D be the polycylinder $|z_1|<1, \cdots, |z_n|<1$ and let G be all automorphisms

$$z_k'=\frac{a_k z_{s_k}+b_k}{\bar{b}_k z_{s_k}+\bar{a}_k},\quad |a_k|^2-|b_k|^2=1,$$

where s_k is a substitution; and let Γ be a subgroup of G with a finite fundamental domain under the usual non-euclidean volume metric. Then if Γ contains some transformation γ where $(\gamma z)_k=z_k$ for all $k\leq\nu$, $(\gamma z)_k\neq z_k$ for $\nu<k\leq n$, it follows that Γ is commensurable with the product of Γ_1 and Γ_2, where for $\gamma\in\Gamma_1$, $(\gamma z)_k=z_k$ $(\nu<k\leq n)$, for $\gamma\in\Gamma_2$, $(\gamma z)_k=z_k$ $(1\leq k\leq\nu)$. The fundamental domain must have finite volume. The proof is based on forming subgroups $G(F_\nu)$ essentially leaving the components $F_\nu=\{(z_1, z_2, \cdots, z_\nu, 1, \cdots, 1)\}$ invariant, and projecting combinations of such subgroups onto Γ.

H. Cohn (Tucson, Ariz.)

3582:

★Malgrange, Bernard. Théorème de Frobenius complexe. Séminaire Bourbaki; 10e année: 1957/1958. Textes des conférences; Exposés 152 à 168; 2e ed. corrigée, Exposé 166, 7 pp. Secrétariat mathématique, Paris, 1958. 189 pp. (mimeographed)

Generalization of the Newlander-Nirenberg proof of a complex Frobenius theorem to a situation involving differential forms.

P. R. Garabedian (New York, N.Y.)

3583:

Nakano, Shigeo. An example of deformations of complex analytic bundles. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 31 (1958), 181–190.

This paper gives an interesting example of the non-Hausdorff phenomenon which one encounters in the theory of deformation of complex structures.

Let T be a complex torus of dimension n, V the symmetric product of T with itself, and S the diagonal on V. Let W be the variety obtained from V by blowing up S, and let X be the inverse image of S in W. The author first shows that X and W are non-singular, and that $X = T \times P_{n-1}$, where P_{n-1} is a complex projective space of dimension $n-1$. Using results of Matsushima [Nagoya Math. J. **14** (1959), 1–24; MR **21** #1403] the author shows that the set of (equivalence classes of) analytic $\mathrm{GL}(2, C)$-bundles over T which have holomorphic connections is parametrized by the non-Hausdorff space $V \cup W$, where $V-S$ is identified with $W-X$. The indecomposable bundles correspond to the points of X, the decomposable ones to the points of V. *M. F. Atiyah* (Cambridge, England)

SPECIAL FUNCTIONS

See also 3731, 3732, 3955, 4017.

3584:

Neville, E. H. The elliptic integrals of the third kind. Canad. J. Math. **11** (1959), 175–194.

L'auteur, qui est aussi celui d'un traité [*Jacobian elliptic functions*, 2ème édition, Clarendon Press, Oxford, 1951; MR **13**, 24], introduit et étudie les intégrales elliptiques qu'il nomme "de 3ème espèce", qui s'écrivent

$$\Pi s(u, a), \quad \Pi s(u, a+K_c), \quad \Pi s(u, a+K_n), \quad \Pi s(u, a+K_d)$$

et qui se définissent par l'intégrale

$$\Pi s(u, a) = \int_0^u \frac{\mathrm{qs}\, a\, \mathrm{qs}'\, a du}{\mathrm{qs}^2 u - \mathrm{qs}^2 a}$$

où K_c, K_n, K_d désignent respectivement les quarts de périodes des fonctions de Jacobi $\mathrm{cs}\, u$, $\mathrm{ns}\, u$, $\mathrm{ds}\, u$.

La quantité $\Pi s(u, a)$ demeure la même lorsque qs désigne successivement l'une de ces 3 fonctions. Les formules relatives à ces fonctions sont données, et utilisent les fonctions de Glaisher. L'intérêt de ces fonctions apparait dans certaines questions relatives à la représentation des fonctions elliptiques habituelles par les fonctions Θ. *R. Campbell* (Caen)

3585:

Tihonov, A. N. The asymptotic behaviour of integrals containing Bessel functions. Dokl. Akad. Nauk SSSR **125** (1959), 982–985. (Russian)

Let V be the class of functions F which are of bounded variation on (λ_0, ∞), where λ_0 may depend upon F, and which are integrable on $(0, \infty)$. Lemma: If F is bounded and belongs to V, then

$$\int_0^\infty I_n(\lambda\rho)F(\lambda)d\lambda = \varepsilon(\rho),$$

where $\varepsilon(\rho) \to 0$ as $\rho \to \infty$. Theorem: If F belongs to V and has n bounded, continuous derivatives on $[0, \infty)$ which also lie in V, then

$$\int_0^\infty J_0(\lambda\rho)F(\lambda)d\lambda = \sum_0^m C_{2k}F^{(2k)}(0)\rho^{-2k-1} + \rho^{-n}\varepsilon(\rho),$$

where $m = [\frac{1}{2}(n-1)]$ and $C_{2k} = (-1)^k(2k-1)!!/2^k k!$.

Sufficient conditions are found under which this asymptotic expansion may be differentiated. *N. D. Kazarinoff* (Madison, Wis.)

3586:

Kogbetliantz, E. G. Sur une identité remarquable concernant la fonction I_0. Chiffres **1** (1958), 121–122.

In this note the author has derived by means of Laplace transformation the following identity:

$$\text{(i)} \qquad e^x f(x, y) + e^y f(y, x) \equiv e^{x+y} - I_0(2\sqrt{(xy)}),$$

$$f(x, y) = \int_0^x e^{-t} I_0(2\sqrt{(yt)})dt,$$

where I_0 denotes the Bessel function with imaginary argument. If I_0 is replaced by the ordinary Bessel function J_0, one finds easily the following identity:

$$\text{(ii)} \qquad e^x f(x, y) - e^{-y} f(y, x) \equiv e^{x-y} - J_0(2\sqrt{(xy)}).$$

N. Chako (Flushing, N.Y.)

3587:

Loh, S. C. On toroidal functions. Canad. J. Phys. **37** (1959), 619–635.

The equation

$$\frac{d^2U}{du^2} + \coth u \frac{dU}{du} - \left(n^2 - \frac{1}{4} + \frac{m^2}{\sinh^2 u}\right)U = 0$$

arises when Laplace's equation is separated in toroidal coordinates. Solutions are

$$p_n^m(u) \equiv P_{n-1/2}^m(\cosh u) \quad \text{and} \quad q_n^m(u) \equiv Q_{n-1/2}^m(\cosh u)$$

(m, n are non-negative integers with $m \leq n$).

The author gives tables of these functions to five figures for $u = 0(0.1)4$, $n = 0(1)4$. For the special case $m = 0$ (zonal toroidal functions) he gives also approximate expressions for u small and u large, graphs, and tables of first derivatives for the same range. *F. M. Arscott* (London)

3588:

Arscott, F. M. A new treatment of the ellipsoidal wave equation. Proc. London Math. Soc. (3) **9** (1959), 21–50.

An ellipsoidal wave function is a solution of

$$\frac{d^2w}{dx^2} - (a + bk^2 \,\mathrm{sn}^2\, z + qk^4 \,\mathrm{sn}^4\, z)w = 0,$$

which is uniform and doubly-periodic in z, its periods being $2K$ or $4K$ and $2iK'$ or $4iK'$. Such solutions exist only for particular values $a(q)$ and $b(q)$; when $q = 0$, they reduce to Lamé polynomials. To find explicit expressions for ellipsoidal wave functions, the following technique is applied. The parameter a is eliminated by setting up the partial differential equation

$$\frac{\partial^2 W}{\partial\alpha^2} - \frac{\partial^2 W}{\partial\beta^2} = \{bk^2(\mathrm{sn}^2\,\alpha - \mathrm{sn}^2\,\beta) + qk^4(\mathrm{sn}^4\,\alpha - \mathrm{sn}^4\,\beta)\}W,$$

which is satisfied by the product of two ellipsoidal wave functions with $z = \alpha$ and $z = \beta$, respectively. W is expanded in a series of ellipsoidal harmonics (Lamé polynomial products). A three term recursion for the coefficients is obtained, whose elements are matrices. The usual technique to solve a three term recursion by continued fractions is generalized by the introduction of continued fractions, whose partial numerators and denominators are matrices. A transcendental equation for the determination of b in terms of q is obtained. Further it is proved that the constructed expansion of W as a series of ellipsoidal harmonics represents an ellipsoidal wave function product.

Thus by giving α or β special values, expansions of ellipsoidal wave functions can be obtained. By using an integral relation, other expansions in terms of Bessel functions are derived, which may be more convenient for practical purposes. The characteristic values of the parameter a are easily obtained by inserting the now known ellipsoidal wave function into its differential equation.

J. Meixner (Aachen)

3589:

Bhonsle, B. R. On a property of generalised Laplace's transformation. Bull. Calcutta Math. Soc. **50** (1958), 6–8.

A theorem is derived which aids the author in the evaluation of certain definite integrals involving the product of a hypergeometric function and MacRobert's E-function.

C. A. Swanson (Vancouver, B.C.)

3590:

Shukla, H. S. A note on the sums of certain bilateral hypergeometric series. Proc. Cambridge Philos. Soc. **55** (1959), 262–266.

In this note, the author gives explicit formulae for the sums of certain bilateral and basic bilateral hypergeometric series. A typical example is

$$ {}_3H_3\left[\begin{matrix} A, & B, & C; \\ 1+C+n, & 1+B, & A-n; \end{matrix}\ 1\right] = \frac{\Gamma(1-C)\Gamma(1-A)\Gamma(1+C+n)}{\Gamma(1-A+n)\Gamma(1+C-B+n)} \times \frac{\Gamma(1-B)\Gamma(1+B)\Gamma(1+B-A+n)}{\Gamma(1+B-A)\Gamma(1+B-C)}. $$

Analogues of this result in terms of basic series are given together with the similar sums and basic analogues for the series

$$ {}_6H_6\left[\begin{matrix} 1+\frac{1}{2}a, & 2+a-c, & c, & d, & e, & f; \\ \frac{1}{2}a, & c-1, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f; \end{matrix}\ -1\right], $$

and

$$ {}_7H_7\left[\begin{matrix} 1+\frac{1}{2}a, & b, & c, & 2+a-c, & e, & f, & g; \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & c-1, & 1+a-e, & 1+a-f, & 1+a-g; \end{matrix}\ 1\right], $$

both of which are well-poised in $1+a$.

L. J. Slater (Cambridge, England)

3591:

Miles, E. P., Jr.; and Williams, Ernest. Basic sets of polynomials for the iterated Laplace and wave equations. Duke Math. J. **26** (1959), 35–40.

Let $n \geqq 0$, $m>0$, $A=\{a=(a_1, a_2, \cdots, a_k) \mid a_i=$ non-negative integer, $a_k \leqq 2m-1$, $\sum a_i=n\}$, and

$$ b_j = (j+[a_k/2])!/j![a_k/2]!(a_k+2j)!. $$

The authors show that the homogeneous polynomial of degree n in k variables,

$$ P_a{}^n = \sum_{j=0}^{[(n-a_k)/2]} \varepsilon^j b_j \Delta^j \left(x_k{}^{a_k+2j} \prod_{i=1}^{k-1} x_i{}^{a_i} \right) $$

is m-harmonic if $\varepsilon=-1$, and satisfies the m times iterated wave equation if $\varepsilon=1$. Furthermore, the set $P=\{P_a{}^n \mid a \in A\}$ is a basic set of such polynomials.

I. I. Kolodner (Albuquerque, N.M.)

3592:

Uvarov, V. B. Theory of a second solution of the differential equation for classical orthogonal polynomials. Dokl. Akad. Nauk SSSR **125** (1959), 281–284. (Russian)

All known properties of second solutions $Q_n(z)$ for differential equations of Jacobi, Laguerre and Hermite orthogonal polynomials are deduced from their common integral representation

$$ \rho(z)\cdot Q_n(z) = \int_a^b (\zeta-z)^{-1}\cdot P_n(\zeta)\rho(\zeta)d\zeta, $$

where $a=-1$, $b=1$, $\rho(z)=(1-z)^\alpha(1+z)^\beta$ in the Jacobi case; $a=0$, $b=\infty$, $\rho(z)=z^\alpha e^{-z}$ in the Laguerre case; and $a=-\infty$, $b=\infty$, $\rho(z)=e^{-z^2}$ in the Hermite case; $P_n(\zeta)$ denoting the corresponding polynomial.

E. Kogbetliantz (New York, N.Y.)

3593:

Chatterjea, S. K. Note on hypergeometric polynomials. Amer. Math. Monthly **66** (1959), 220–223.

The following formula is proved for the ultraspherical polynomials [Thiruvenkatachar and Nanjundiah, Proc. Ind. Acad. Sci. Sect. A **33** (1951), 373–384; MR **14**, 44]:

$$ (1-x^2)\{[DP_n]^2-[DP_{n-1}][DP_{n+1}]\} = n(n+2\lambda)[P_n]^2-(n+1)(n+2\lambda-1)P_{n-1}P_{n+1}; $$
$$ P_n = P_n{}^{(\lambda)}, \quad D = d/dx. $$

Moreover the following identity is derived:

$$ (1-2ah+h^2)^{-\alpha}(1-2bh+h^2)^{-\beta} = \frac{2}{B(\alpha,\beta)} \int_0^{\pi/2} \frac{(\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}d\theta}{\{1-2(a\sin^2\theta+b\cos^2\theta)h+h^2\}^{\alpha+\beta}}, $$

where B is Euler's integral. Several consequences are pointed out.

G. Szegö (Stanford, Calif.)

3594:

Danese, Arthur E. Some identities and inequalities involving ultraspherical polynomials. Duke Math. J. **26** (1959), 349–359.

Let $P_n{}^\lambda(x)$ denote the ultraspherical polynomial of degree n and put

$$ p_n{}^\lambda = P_n{}^\lambda(x)/P_n{}^\lambda(1). $$

The paper contains numerous identities and inequalities involving the combination

$$ \phi_{n+r}(x)\phi_{n+s}(x)-\phi_n(x)\phi_{n+r+s}(x), $$

where $\phi_n(x)$ is taken to be $p_n{}^\lambda(x)$, $P_n{}^\lambda(x)$ or $(d^r/dx^r)P_n{}^\lambda(x)$. For example, it is proved that for $\lambda>0$, $0 \leqq x \leqq 1$,

$$ p_{n+1}^\lambda(x)p_{n+2}^\lambda(x)-p_n^\lambda(x)p_{n+3}^\lambda(x) \geqq 0, $$

with equality only for $x=0$ or $x=1$. Some inequalities for the Jacobi polynomials

$$ p_n{}^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n}^{-1} P_n{}^{(\alpha,\beta)}(x) $$

are obtained in the special case $\beta=\alpha+1$.

L. Carlitz (Durham, N.C.)

3595:

Rajagopal, A. K. A note on the generalisation of Hermite polynomials. Proc. Indian Acad. Sci. Sect. A. **48** (1958), 145–151.

Der Verfasser studiert die, als Verallgemeinerung der Hermiteschen Polynome von E. T. Bell [Ann. of Math. (2) **35** (1934), 258–277] eingeführten, Polynome $\xi_n(x, t, r) = \exp(xt^r)(\partial/\partial t)^n \exp(-xt^r)$ und gibt, mittels eines Lemmas von Dave Pandres (1957) eine Determinantendarstellung, die für den Spezialfall der H. Polynome eine von der Nielsen'schen Darstellung (1926) abweichende, neue Darstellung der H. Polynome ergibt. Anwendung der Truesdell'schen F. Gleichung: $\partial F(t, \alpha)/\partial t = F(t, \alpha+1)$ und einiger anderer Resultate von Truesdell liefert weitere Funktionalgleichungen der Polynome $\xi_n(x, t, r)$.

K. Endl (Columbus, Ohio)

3596:

Carlitz, L. Eulerian numbers and polynomials. Math. Mag. **32** (1958/59), 247–260.

This is mainly an expository article. The author gives a connected account of many properties of the numbers H_n, given by

$$\frac{1-\lambda}{e^x-\lambda} = \sum_{n=0}^{\infty} H_n \frac{x^n}{n!} \quad (\lambda \neq 1),$$

and the related polynomials

$$H_n(u|\lambda) = \sum_{r=0}^{n} \binom{n}{r} H_r u^{n-r}.$$

His main object is to indicate their connection with Bernoulli numbers and polynomials, and to obtain some congruential properties of H_n. A good bibliography is given, and it is pointed out that Eulerian numbers and polynomials have found application in certain problems of combinatorial analysis and in the devising of criteria for Fermat's last theorem.

H. Halberstam (London)

3597:

Al-Salam, W. A.; and Carlitz, L. Bessel polynomials and Bernoulli numbers. Arch. Math. **9** (1958), 412–415.

The Bessel polynomials, which can be defined by

$$y_{n+1}(x) = (2n+1)xy_n(x) + y_{n-1}(x),$$
$$y_0(x) = 1, \quad y_1(x) = x+1,$$

were introduced by H. L. Krall and O. Frink [Trans. Amer. Math. Soc. **65** (1949), 100–115; MR **10**, 453]. The authors also consider $u_n(x)$, defined by the same recurrence formula, with the initial conditions $u_0(x)=1$, $u_1(x)=x$. And they put

$$v_n(x) = \tfrac{1}{2}(y_{n+1}(x) + (-1)^n y_{n+1}(-x)).$$

Both $u_n(x)$ and $v_n(x)$ are of degree n, and are even if n is even, odd if n is odd. In another paper [Duke Math. J. **26** (1959), 437–445; MR **21** #4256] the authors show connections with Bernoulli and Euler numbers, and establish orthogonality of $\{u_n\}$, and also of $\{v_n\}$ on the unit circle, with appropriate weight-functions.

In the present paper they show that $\{u_n\}$ is an orthogonal system on the imaginary axis, with weights localized to the points $2((2k+1)\pi i)^{-1}$ $(k=0, \pm 1, \cdots)$. A similar result is given for $\{v_n\}$. As an application a Hankel determinant involving Bernoulli numbers and a similar determinant with Euler numbers are evaluated.

N. G. de Bruijn (Amsterdam)

ORDINARY DIFFERENTIAL EQUATIONS

See also 3526, 3645, 3664, 3667, 3668, 3954, 4022, 4063, 4064.

3598:

Picone, Mauro; e Gross, Wolf. Intervallo d'esistenza e limitazioni per la soluzione di un sistema di equazioni differenziali ordinarie. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. **25** (1958), 225–230.

The authors establish an existence, uniqueness and boundedness theorem for a system of ordinary differential equations satisfying certain special conditions. The method is that of successive approximations.

R. Bellman (Santa Monica, Calif.)

3599:

Bihari, Imre. On the unicity of the solutions of the ordinary first order differential equations. Mat. Lapok **8** (1957), 115–119. (Hungarian. English and Russian summaries)

The author gives conditions under which if $u' = \omega(x, u)$ has more than one solution passing through the origin, then so does $y' = f(x, y)$; here f, ω are continuous and $\omega \geqq 0$. Among other things, it is required that $\omega(x, 0) = 0$, and that

$$|f(x, y) - f(x, \phi)| \geq \omega(x, |y-\phi|),$$

where $\phi(x)$ is some solution of $y' = f(x, y)$ with $\phi(0) = 0$.

F. V. Atkinson (Canberra)

3600:

Grinfel'de, A. R. Application of the method of upper and lower functions to the solution of a non-linear differential system of second order. Latvijas Valsts Univ. Zinātn. Raksti **20** (1958), no. 3, 57–63. (Russian. Latvian summary)

The paper contains a slight extension of results of Babkin [Akad. Nauk SSSR. Prikl. Mat. Meh. **18** (1954), 239–242; MR **15**, 793] on successive upper and lower approximations which converge monotonically to the solution of $y'' = f(x, y)$, $y(a) = A$, $y(b) = B$, when $\partial f/\partial y \geqq 0$.

J. P. LaSalle (Baltimore, Md.)

3601:

Čandirov, G. I. A generalization of the inequality of Gronwall and its applications. Azerbaĭdžan. Gos. Univ. Uč. Zap. **1958**, no. 6, 3–11. (Russian. Azerbaidzani summary)

The inequality of the title is as follows: Let $b(x)$ be continuous on $0 \leqq x \leqq l$ and satisfy the inequality

$$0 \leq b(x) \leq \int_0^x [a(t)b(t) + c(t)]dt + f(x),$$

where $a(x)$ and $c(x)$ are non-negative integrable functions on the given interval, while $f(x)$ is bounded. Then

$$(1) \qquad b(x) \leqq \left[\int_0^x c(t)dt + \sup|f|\right] \exp\left(\int_0^x a(t)dt\right).$$

Consider the differential equation (2) $y' = f(x, y)$, where $f(x, y)$ is measurable in x, and Lipschitzian in y with respect to a function $\beta(x)$ which is summable on $[0, l]$. By using (1) together with the usual method of successive approximations, the author makes estimates of the speed of convergence of the approximations to the solution, and

of the amount of perturbation in the solution caused by a change in the initial conditions, etc. The second portion of the paper is concerned with a "mixed" boundary and initial value problem for the equation $u_{tt}=Au+F(t, x, u)$, where A is a self-adjoint operator which is half-bounded from above, with suitable restrictions on its eigenvalues and eigenfunctions. The function $F(t, x, u)$ is assumed to be Lipschitzian in u with respect to a function $a(t, x)$ which lies in L^2. Again the inequality (1) is used to provide an estimate of the difference between a certain approximate solution and the exact solution of this mixed problem.

D. H. Hyers (Los Angeles, Calif.)

3602:

Karasev, I. M. Integration of a type of degenerate hypergeometric equation. Kabardin, Gos. Ped. Inst. Uč. Zap. **12** (1957), 29–38. (Russian)

Conditions are given under which

$$(a_2+b_2x)y''+(a_1+b_1x)y'+(a_0+b_0x)y = 0$$

has a solution of the form $e^{\lambda x}D^n[e^{\mu x}(a_2+b_2x)^k]$ where D is the usual differential operator.

R. G. Langebartel (Urbana, Ill.)

3603:

Klamkin, Murray S.; and Newman, Donald J. On the reducibility of some linear differential operators. Amer. Math. Monthly **66** (1959), 293–295.

The authors prove $x^nD^{2n}=[xD^2-(n-1)D]^n$ and $x^{2n}D^n=[x^2D-(n-1)x]^n$, where $D=d/dx$ (and point out that the second of these relations is equivalent to a result of Glaisher [Nouvelle Corr. Math. **2** (1876), 240–243, 349–350]); and they use these relations to solve the differential equations $x^nD^{2n}y=y$, $x^{2n}D^ny=y$, and $D^2x^4D^2u=x^2u$. They also give some generalizations of their identities and solve some further differential equations.

A. Erdélyi (Pasadena, Calif.)

3604:

Matsumoto, Katsumasa. A new deduction of singularity criteria for the first order differential equations. Kumamoto J. Sci. Ser. A **3**, 87–92 (1957).

The author presents a new method for analyzing the differential equation

$$\frac{dy}{dx}=\frac{\alpha x+\beta y}{\gamma x+\delta y}$$

near $x=y=0$, claiming that his procedure is clearer and simpler than the traditional one. The arguments appear neither clear nor simpler to this reviewer.

W. Wasow (Madison, Wis.)

3605:

Guderley, Karl G. Asymptotic representations for differential equations with a regular singular point. Arch. Rational Mech. Anal. **3**, 206–218 (1959).

The author considers the equation

$$y''-\lambda^2x^k(x-1)^{-2}g_1(x, \lambda^{-1})y = 0$$

with $k=0$ or 1, where g_1 is holomorphic and free from singularities and zeros wherever necessary. He shows that if

$$g_2(x, \lambda^{-1}) = [g_1(x, \lambda^{-1})+\tfrac{1}{4}\lambda^{-2}]^{1/2},$$

$$h_k(x) = e^{i\pi}\int_0^x (t-1)^{-1}t^{k/2}g_2(t, \lambda^{-1})dt,$$

with the branch of g_2 and arg t as $t\to 0$ suitably defined, then $Y_{1,2}=h_0'^{-1/2}\exp(\mp\lambda h_0)$ are two linearly independent solutions for $k=0$ which are asymptotic in the following sense. If y is an exact solution such that at some point $x=b$

$$y = Y_j[1+O(\lambda^{-1})], \quad y' = Y_j'[1+O(\lambda^{-1})]$$

where $j=1$ or 2, then these relations hold for all points that can be reached along paths for which Re $(\pm\lambda h_0)$ does not decrease. Similarly,

$$\overline{Y}_j = e^{2i\pi/3+(2-j)i\pi/2}\lambda^{1/2}\left(\frac{\pi}{2}\right)^{1/2}h_1'^{-1/2}h_1^{1/2}H_{1/3}^{(1)}(\lambda h_1e^{i\pi/2+(2-j)i\pi})$$

are linearly independent solutions for $k=1$ such that if, at some point $x=b$, $y_j(b)=\overline{Y}_j(b)$, $y_j'(b)=\overline{Y}_j'(b)$ then

$$y_j = \overline{Y}_j+O(\lambda^{-5/6})|h_1|^{1/6}|h_1'|^{-1/2}M(N+|\lambda|^{2/3}|h_1|^{2/3})^{-1/4}e^{\pm\lambda h_1}$$

for all points that can be reached along paths for which $\mathrm{Re}(\pm\lambda h_1)$ does not decrease; M, N are positive constants independent of x and λ, and arg λh_1 is suitably restricted.

H. A. Antosiewicz (Los Angeles, Calif.)

3606:

★Fazekas, F.; und Sándor, I. Über die Lösung der gewöhnlichen linearen inhomogenen Differentialgleichung zweiter Ordnung. Einige Arbeiten des Lehrstuhles für Mathematik im Lehrjahre 1956/57, pp. 75–79. Wissenschaftliche Veröffentlichungen der Technischen Universität für Bau- und Verkehrswesen in Budapest, Budapest, 1958. 80 pp.

For the ordinary linear second order differential equation

$$y''+p(x)y'+q(x)y = f(x)$$

the authors present some elementary solution formulas of well-known character. However, they fail to point out that certain of the given formulas are valid only for intervals I on which the homogeneous equation $y''+p(x)y'+q(x)y=0$ is non-oscillatory in the sense that there exists a solution of this latter equation which is non-zero throughout I.

W. T. Reid (Iowa City, Iowa)

3607:

Karamyshkin, V. V. Transformation of a linear differential equation with polynomial coefficients into an integral equation with the aid of operational calculus. J. Appl. Math. Mech. **22** (1958), 774–776 (553–554 Prikl. Mat. Meh.).

With the aid of elementary properties of the Laplace transformation the author points out how a class of ordinary differential equations with polynomial coefficients can be written as integral equations of Volterra type.

R. V. Churchill (Ann Arbor, Mich.)

3608:

Borůvka, O. Théorie analytique et constructive des transformations différentielles linéaires du second ordre. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine **1 (49)** (1957), 125–130.

Es handelt sich um folgendes noch von Kummer herrührendes Problem: Wenn zwei Differentialgleichungen von sog. Jacobischem Typus (1) $y''=q(t)y$; (2) $Y''=Q(T)Y$ und ein Integral $U(T)$ der Gleichung (2) gegeben sind, wobei q

und Q kontinuierliche Funktionen sind, zwei Funktionen $w(t)$ und $X(t)$ derart zu finden, dass $u(t)=w(t)\cdot U[X(t)]$ ein Integral der Gleichung (2) wird. Zu dem Zweck entwickelt der Verf. eine Theorie der linearen Differentialtransformationen von zweiter Ordnung, die aus einem analytischen und einem sog. konstruktiven Teil besteht. In der vorliegenden Arbeit wird der analytische Teil ganz kurz gestreift, da er ausführlich vom Verf. früher veröffentlicht wurde [Ann. Mat. Pura Appl. (4) **41** (1956), 325–342; MR **20** #1814], während der konstruktive Teil etwas vollständiger behandelt wird. *T. P. Anđelić* (Belgrade)

3609:

Hartman, Philip. On the ratio $f(t+cf^{-\alpha}(t))/f(t)$. Boll. Un. Mat. Ital. (3) **14** (1959), 57–61.

"The condition

$$(1)\qquad f(t+cf^{-\alpha}(t))/f(t)\to 1 \quad\text{as}\quad t\to\infty,$$

its differentiated form

$$(2)\qquad f'(t)/f^{1+\alpha}(t)\to 0 \quad\text{as}\quad t\to\infty$$

and its integrated form

$$(3)\quad \log(f(u)/f(v))\Big/\left(1+\int_u^v f^{\alpha}(s)ds\right)\to 0 \quad\text{as}\quad u,v\to\infty$$

frequently occur in the asymptotic integration theory of $d^2x/dt^2\pm f(t)x=0$. The results proved imply that (3) and $\int^\infty f^{\alpha}(t)dt=\infty$ are equivalent to (1)." (Author's summary)
J. K. Hale (Baltimore, Md.)

3610:

Hartman, Philip. On oscillators with large frequencies. Boll. Un. Mat. Ital. (3) **14** (1959), 62–65.

Another proof is given for the theorem of Armellini, Sansone and Tonelli [see G. Sansone, *Equazioni differenziali nel campo reale*, vol. 2, Zanichelli, Bologna, 1949; MR **11**, 32] that if $q(t)$ is continuous for $t\geqq 0$ and is monotone, $q(t)\to\infty$ as $t\to\infty$, and if $\log q(t)$ is of "regular growth", then all solutions of $d^2x/dt^2+q(t)x=0$ tend to zero as $t\to\infty$. The proof, by contradiction, is very simple and avoids the use of Sturm's second comparison theorem. The author states that his method of proof can be modified to include the more general case considered by Opial [Ann. Polon. Mat. **5** (1958), 77–93; MR **20** #4047].
J. K. Hale (Baltimore, Md.)

3611:

Bellman, Richard. On the non-negativity of Green's functions. Boll. Un. Mat. Ital. (3) **12** (1957), 411–413.

In this note the author proves a special case of a result of Aronszajn and Smith [Tech. Rep. **15**, Univ. of Kansas, 1956; Amer. J. Math. **79** (1957), 611–622; MR **19**, 566]. The main theorem is that the Green's function for the boundary value problem (1) $u''+q(x)u=f(x)$, $u(0)=u(1)=0$, is non-positive if, for $0\leqq x\leqq 1$, $q(x)\leqq\pi^2-d$ for some $d>0$. The proof, which uses the variational problem associated with (1), extends, as the author points out, to the case of other boundary conditions if the condition on $q(x)$ is replaced by $q(x)\leqq\lambda_1-d$ for some $d>0$ where λ_1 is the least eigenvalue of the Sturm-Liouville problem $u''+\lambda u=0$ with the given boundary condition. The author also remarks that with the same proof and similar conditions, one can obtain the same result for $\Delta u+qu=f$ with appropriate types of boundary conditions.
R. R. D. Kemp (Kingston, Ont.)

3612:

Grunsky, Helmut. Ein nichtlineares Randwertproblem im Komplexen. Math. Nachr. **19** (1958), 255–264.

The author discusses the determination in the complex plane of solutions of an analytic differential equation $w''=F(z,w)$ subject to two boundary conditions $w(z_1)=w(z_2)=0$. The analysis hinges on the formula

$$2Aw(z)=(z-z_1)(z-z_2)\iint w''(\zeta)d\xi d\eta,$$

where the double integral is taken over the triangle with vertices at z, z_1, and z_2, and where A stands for the area of this triangle. Conditions for solving the above boundary value problem in a convex region are given in terms of related eigenvalues. *P. R. Garabedian* (New York, N.Y.)

3613:

Èĭdus, D. M. Some inequalities for eigenfunctions. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 796–798. (Russian)

3614:

Opial, Z. Sur la répartition asymptotique des zéros des fonctions caractéristiques du problème de Sturm. Ann. Polon. Math. **6** (1959/60), 105–110.

The equation of Sturm is (1) $(p(x)u')'+(\rho(x)\lambda+q(x))u=0$, where p, ρ, and q are continuous on a closed interval $[a,b]$ and p and ρ are positive. Let $\lambda_1<\lambda_2<\cdots$ be the characteristic values of (1)—i.e., each λ_n is such that for $\lambda=\lambda_n$ (1) has a unique non-trivial solution $u_n(x)$ satisfying the conditions $u(a)=u(b)=0$, $\int_a^b u_n^2(x)dx=1$, $u_n'(a)>0$. The existence of such characteristic values and the facts that as $n\to\infty$, $\lambda_n\to\infty$, and that $u_n(x)$ has exactly $n+1$ zeros in the interval $[a,b]$ are conclusions of the classical theorems of Sturm [Courant and Hilbert, *Methods of mathematical physics*, vol. I, Interscience, New York, 1953; MR **16**, 426]. In this note the author poses the following question: if $G_n(\alpha,\beta)$ denotes the number of zeros of $u_n(x)$ in a closed subinterval $[\alpha,\beta]$ of $[a,b]$, does the limit of the quotient $G_n(\alpha,\beta)G_n^{-1}(a,b)=G_n(\alpha,\beta)(n+1)^{-1}$ exist as $n\to\infty$? In other words, how are the zeros of the characteristic functions distributed among the subintervals of $[a,b]$? The author's answer is that under the above assumptions on p, ρ, and q, this limit exists and is in fact

$$\left(\int_\alpha^\beta \rho(x)^{1/2}p(x)^{-1/2}dx\right)\cdot\left(\int_a^b \rho(x)^{1/2}p(x)^{-1/2}dx\right)^{-1}.$$

The proof is not difficult and uses the asymptotic formulas for the characteristic functions [ibid.]. In conclusion the author states that a similar method would work for more complicated boundary conditions.
C. S. Coleman (Claremont, Calif.)

3615:

Opial, Z. Nouvelles remarques sur l'équation différentielle $u''+a(t)u=0$. Ann. Polon. Math. **6** (1959/60), 75–81.

The present paper is a continuation of the author's work [same Ann. **5** (1958), 77–93, MR **20** #4047] on the equation $u''+a(t)u=0$, where $a(t)$ is a continuous function of t, $0\leqq t<+\infty$ and $a(t)\to\infty$ monotonically as $t\to\infty$. Theorem: If $a(t)$ is as above, $q(t)$ is a continuous function such that $\int_0^\infty[a(t)]^{-1/2}|q(t)|dt<\infty$, then all solutions of (1) $u''+[a(t)+q(t)]u=0$ are bounded for $t\geqq 0$ and there exists at least one solution which approaches zero as

$t\to\infty$. Also, for any solution $u(t)$ of (1), the function $A(t)=[u^2(t)+u'^2(t)a^{-1}(t)]^{1/2}$ is of bounded variation in $[0,+\infty)$. Similar results are also given for the more general equation $u''+[a(t)+\psi(t)+q(t)]u=0$ with a, q as above and $\psi(t)$ of bounded variation in every finite interval. *J. K. Hale* (Baltimore, Md.)

3616:

Dunford, Nelson. A survey of the theory of spectral operators. Bull. Amer. Math. Soc. **64** (1958), 217–274.

This is a survey of recent results due primarily to a group of mathematicians centered around the author and anticipating the publication of the second volume of Dunford and Schwartz, *Linear Operators* [vol. I, Interscience, New York, 1958]. A determined effort went into finding classes of, especially differential, operators which are spectral operators in the author's sense. D. McGarvey proved that a large class of nth order differential operators that are singular and with periodic coefficients have a countable additive spectral measure. This seems to be so far an unpublished result. Earlier are results by Schwartz and Kramer on regular nth order differential boundary problems with sufficiently regular boundary conditions. Results by M. A. Naĭmark [Uspehi Mat. Nauk (N.S.) **8** (1953), no. 4 (56), 174–175; Dokl. Akad. Nauk SSSR **89** (1953), 213–216; Trudy Moskov. Mat. Obšč. **3** (1954), 181–270; MR **15**, 530, 30, 959] have been enlarged by Schwartz. K. O. Friedrichs has proved [Comm. Pure Appl. Math. **1** (1948), 361–406; MR **10**, 547] that under certain conditions $Tf(s)=sf(s)+\int_a^b K(s,t)f(t)dt$ is a scalar operator. The conditions are either of the Lipschitz type on K or conditions on its Fourier transform. To be noted is a conjecture of Schwartz that the spectrum of a singular differential operator under quite general conditions on the rate of growth of the coefficients and quite general boundary conditions will consist of a finite or enumerable number of analytic arcs running into and out of a finite or enumerable set of branch points together with an enumerable or vacuous set of point eigenvalues whose only limit points are the branch points. It is further conjectured that the resolvent will have first order rate of growth along the normals to points interior to the analytic arcs and that the corresponding operators will be spectral in the complement of that portion of L^2 which is associated with arbitrary small neighborhoods of the branch points. The next chapter reviews properties of spectral operators and substantially repeats the author's papers in Pacific J. Math. **2** (1952), 559–614; **4** (1954), 321–354 [MR **14**, 479; **16**, 142]. Wermer has shown [ibid. **4** (1954), 355–361; MR **16**, 143] that scalar operators in Hilbert space are those similar to normal operators. Bade [ibid. **4** (1954), 373–392; MR **16**, 143] developed the theory of unbounded spectral operators. In other papers Bade [Trans. Amer. Math. Soc. **80** (1955), 345–360; MR **17**, 513; and a paper to appear in Trans. Amer. Math. Soc. in the early part of 1960] developed a theory of algebras of spectral operators extending many features of commutative W^*-algebras. Foguel [Comm. Pure Appl. Math. **11** (1958), 293–295; MR **20** #3458] has shown that for the strong limit of a sequence of commuting spectral operators the spectral measure is the limit of the spectral measures if the Boolean algebra generated by the spectral measures of the terms of the sequence is bounded. In the next chapter the author reviews conditions on the resolvent of an operator which are sufficient to insure that it is a spectra operator. These ideas have been developed explicitly in Dunford [loc. cit.]. The proofs are given again in a somewhat improved form. In the last chapter special attention is paid to operators having spectra in Jordan curves. The results are again contained in previous papers, but they are improved in many directions. Special attention is also paid to some results of Naĭmark, which have been mentioned above. *František Wolf* (Berkeley, Calif.)

3617:

Kondrat'ev, V. A. Extensions of linear differential operators. Dokl. Akad. Nauk SSSR **125** (1959), 479–481. (Russian)

The deficiency indices and location of essential spectrum are computed for a class of differential operators in Hilbert space $L_2(0,\infty)$ [for definitions, see G.-C. Rota, Comm. Pure Appl. Math. **11** (1958), 23–65; MR **20** #3334], namely the operators

$$L(y)=(p_0y^{(n)})^{(n)}+(p_1y^{(n-1)})^{(n-1)}+\cdots+p_ny,$$

where the p_i are complex-valued functions continuous in $[0,\infty)$. Theorem 1: The deficiency indices in any connected component of the essential resolvent set are equal. Writing $L(y)=L_1(y)+iL_2(y)$, where the L_1 and L_2 are formally symmetric differential operators. Theorem 2: If the essential spectrum of L_1 is the empty set and if L_1 is bounded below, then the spectrum of every extension of L is either the entire plane or discrete. Applications are made to show that the essential spectra of the following operators are the empty set: $(-1)^{(n)}y^{(2n)}+q(x)y+ip(x)y$ with $\lim_{x\to\infty}\cdot Aq(x)+Bp(x)=+\infty$; L_1 is of the same form as L with $p_0>M>0$ and $(-1)^{n-k}p_k$ bounded below, and $p_n(x)\to\infty$, or else $L_2(y)=q(x)y$ with $\lim_{x\to\infty}|q(x)|=\infty$. *G.-C. Rota* (Cambridge, Mass.)

3618:

Kemp, R. R. D.; and Levinson, Norman. On $u''+(1+\lambda g(x))u=0$ for $\int_0^\infty|g(x)|dx<\infty$. Proc. Amer. Math. Soc. **10** (1959), 82–86.

It is known that equations of the type indicated in the title possess asymptotic solutions of the form $r(\lambda)\sin(x+\theta(\lambda))$ as $x\to\infty$. Answering a question of the reviewer, the authors determine the analytic behavior of the functions $r(\lambda)$ and $\theta(\lambda)$ as functions of the complex variable λ. *R. Bellman* (Santa Monica, Calif.)

3619:

★**Mendelson, Pinchas. On phase portraits of critical points in n-space.** Contributions to the theory of nonlinear oscillations, Vol. IV., pp. 167–199. Annals of Mathematics Studies, no. 41. Princeton University Press, Princeton, N.J., 1958. ix+211 pp. \$3.75.

The author studies the nature of an isolated singular point at the origin for a real vector differential equation $\dot{x}=Ax+R(x)$, where $x=(x_0,x_1,\cdots,x_n)$, A is a constant $(n+1)$ by $(n+1)$ matrix of rank n, with one zero characteristic root, all other characteristic roots having nonzero real parts of the same sign; $R(x)$ is assumed analytic at the origin and containing no constant or linear terms. The equation is reduced to a canonical form in which the first component of x satisfies an equation $\dot{x}_0=x_0^k+P_0(x)$,

where $k \geqq 2$ and $P_0(x)$ has terms only of degree d or higher, there being no terms in x_0 alone. A detailed study of the configurations, illustrated by 16 figures, is carried out under the further assumption that $d=k$.

W. Kaplan (Ann Arbor, Mich.)

3620:

Ezeilo, J. O. C. On the boundedness of solutions of a certain differential equation of the third order. Proc. London Math. Soc. (3) **9** (1959), 74–114.

The author gives two boundedness theorems for the third order equation

$$\dddot{x}+a\ddot{x}+b\dot{x}+f(x) = p(t), \tag{1}$$

a and b constants. These results are of the same type as the ones obtained for second order equations by M. L. Cartwright [*Contributions to the theory of nonlinear oscillations*, pp. 149–241, Princeton Univ. Press, Princeton, N.J., 1950; MR **11**, 722]. One of the author's theorems is the following. Let $|f(x)| \leqq M < \infty$ for all x and assume also that: $a>0$, $b>0$; $f(x) \operatorname{sgn} x \geqq m > 0$ for $|x| \geqq 1$; $|p(t)| \leqq A$, $|\int_0^t p(\tau)d\tau| \leqq A$ for all t. Then there exists $D>0$ depending on a, b, A and M such that every solution $x(t)$ of (1) is such that $x^2+\dot{x}^2+\ddot{x}^2 \leqq D$ for all t large enough. The proof is too complicated to be sketched here.

M. M. Peixoto (Baltimore, Md.)

3621:

Krasnosel'skiĭ, M. A.; and Perov, A. I. On a certain principle of existence of bounded, periodic and almost periodic solutions of systems of ordinary differential equations. Dokl. Akad. Nauk SSSR **123** (1958), 235–238. (Russian)

The authors consider an n-dimensional system $x'=f(t,x)$, where f is continuous for $-\infty<t$, $\|x\|<+\infty$. Suppose that there exist two continuously differentiable functions λ, μ on $\|x\|<\infty$ such that $\sum_{i=1}^n f_i(\partial\lambda/\partial x_i)>0$ for $\|x\| \geqq R$, where R is some positive number, and $\sum_{i=1}^n f_i[\partial(\lambda+\mu)/\partial x_i] \geqq 0$ on the set T of all $x \in E_n$ for which $m \leqq \lambda(x) \leqq M$, $\|x\|>R$. Here $m=\min \lambda(x)$, $M=\max \lambda(x)$, for $\|x\|=R$. Moreover let $|\mu(x)| \to \infty$ as $x \in T$, $\|x\| \to \infty$. A system for which such functions λ, μ exist is said to be of type (*). If the system is of type (*), and $\lambda(-x)=\lambda(x)$, then there exists at least one solution which is bounded on $-\infty<t<\infty$; if f is periodic in t there exists at least one periodic solution with the same period as f; if f is almost periodic in t (uniformly in each sphere) there is at least one almost periodic solution.

A function λ is said to be non-degenerate if its gradient does not vanish outside some sphere and if grad λ has a non-zero rotation on spheres of sufficiently large radii with centers at the origin. If the system is of type (*) with a non-degenerate λ, the same conclusions as mentioned above can be drawn. It is mentioned that the proofs depend on a simple topological fact. Examples are given to illustrate the results.

E. A. Coddington (Los Angeles, Calif.)

3622:

Chin, Yuan-shun. Sur les équations différentielles à la surface du tore. I. Acta Math. Sinica **8** (1958), 348–368. (Chinese. French summary)

Part I deals with a one-parameter family of equations

$$\frac{d\theta}{d\phi} = A(\theta, \phi; \lambda),$$

A of period 2π in θ and ϕ. The dependence of the Poincaré rotation number μ on the parameter λ is examined in the case when A is strictly monotone increasing in λ. $\mu(\lambda)$ is then monotone, but not always strictly. Theorem 1: If $\lambda_2<\lambda_1$ and at least one of the numbers $\mu(\lambda_2)$, $\mu(\lambda_1)$ is irrational, then $\mu(\lambda_2)<\mu(\lambda_1)$. Other theorems in this section relate the graph $(\lambda, \mu(\lambda))$ to the existence of limit cycles and stability. Part II discusses the equation under smoothness hypotheses and positive righthand side. An explicit formula is given for μ in case $A(\theta,\phi)=\Theta(\theta,\phi)/\Phi(\theta,\phi)$ and the system has integrating factor $M(\theta,\phi)$, namely,

$$\mu = \left[\int_0^{2\pi}\int_0^{2\pi} M\Theta d\theta d\phi\right] \Big/ \left[\int_0^{2\pi}\int_0^{2\pi} M\Phi d\theta d\phi\right].$$

{This formula seems to appear only in the resumé. But it is implicit in Schwartzman, Ann. of Math. (2) **66** (1957), 270–284 [MR **19**, 568].} Finally, in Part III, specific equations—Θ, Φ trigonometric polynomials—are examined in detail. (From the author's summary)

L. W. Green (Minneapolis, Minn.)

3623:

Langenhop, C. E.; and Seifert, G. Almost periodic solutions of second order nonlinear differential equations with almost periodic forcing. Proc. Amer. Math. Soc. **10** (1959), 425–432.

The equations considered are real and of the form

$$x''+f(x)x'+g(x) = ke(t), \quad x' = dx/dt,$$

wherein $e(t)$ is continuous, almost periodic and $|e(t)| \leqq 1$. Setting $F(x)=\int_0^x f(v)dv$, the authors consider the system $x'=y-F(x)$, $y'=-g(x)+kp(t)$, wherein $k>0$ and $p(t)$ is any element of the closure, with respect to the uniform norm, of the set $\{e(t+h): -\infty<h<\infty\}$. Concerning g, f, F they assume that there exist positive real constants a, b, c, d, α, β such that $c<d$, $a>b$ and (A) $g(c)=k$, $g(-b)=-k$; (B) for $-a \leqq x \leqq d$: (i) $k<(F(d)-F(c))f(x)+g(-a)$; (ii) $k<(F(-b)-F(-a))f(x)-g(d)$; (iii) $g(x)$ has a continuous derivative $g'(x)$, $g(0)=0$, and $0<g'(x) \leqq \beta$ for $x \neq 0$; (iv) $f(x) \geqq \alpha$; (v) $\beta<\alpha^2$. Under these assumptions, there exists an almost periodic solution of the system of differential equations and the trajectory of this solution is contained in a region R such that the solution is asymptotically stable with respect to all solutions whose trajectories enter R.

This theorem is compared with almost periodicity theorems of L. Amerio [Ann. Mat. Pura Appl. **39** (1955), 97–119; MR **18**, 128], G. I. Biryuk [Dokl. Akad. Nauk SSSR **97** (1954), 577–579; MR **16**, 249], J. J. Schaffer [Rend. Circ. Mat. Palermo **5** (1956), 204–236; MR **18**, 576] and G. E. H. Reuter [J. London Math. Soc. **26** (1951), 215–221; MR **13**, 237].

W. R. Utz (Columbia, Mo.)

3624:

Golomb, Michael. Expansion and boundedness theorems for solutions of linear differential systems with periodic or almost periodic coefficients. Arch. Rational Mech. Anal. **2** (1958), 284–308.

The author considers the general linear system of equations

$$Kdw/dz = i[Aw+\lambda B(z)w], \tag{1}$$

where $i=\sqrt{-1}$, K, A are constant $n\times n$ matrices, z is a real or complex variable and

$$(2)\qquad B(z) = B_0+\sum_{k\geq 1} B_k \exp(i\rho_k z),$$

with B_k ($k=0, 1, \cdots$) constant matrices and the ρ_k complex numbers. Solutions of (1) of the form

$$w(z) = \exp(i\tau z)\sum_{r\geq 1} c_r \exp(i\omega_r z)$$

are given, where the ω_k are linear combinations of the ρ_k with non-negative integral coefficients. Explicit expansions are obtained for both the coefficients c_r and the determining equation whose roots are the characteristic exponents τ. Let $\sigma_1, \cdots, \sigma_m$ ($m\leq n$) be the zeros of $\det(\sigma K-A)$. The series solution of the above type is shown to converge provided that (2) is absolutely convergent and $\inf|\sum_{k\geq 1} n_k\rho_k+\sigma_\nu-\sigma_\mu|>0$, where ν, $\mu=1, 2, \cdots, m$, and the n_k ($k=1, 2, \cdots$) are non-negative integers such that only a finite number are $\neq 0$ and $\sum n_k\rho_k\neq 0$. The method used is similar to the one used by the author [#3625 below] and W. Wasow [Illinois J. Math. **2** (1958), 254–260; MR **20** #2526] in connection with nonlinear systems having small forcing terms. These results are then applied to the difficult case where the σ_μ are real to obtain criteria for the boundedness and asymptotic stability of the solutions of (1). Some of the results for the case where z is a real variable and $B(z)$ is periodic in z are stated below. Consider the system of second order equations

$$(3)\qquad x''+Ax = \lambda\Phi(z)x,$$

where $x=(x_1, \cdots, x_n)$, $A=\operatorname{diag}(\alpha_1^2, \cdots, \alpha_n^2)$, and

$$\Phi(z) = \Phi_0+\sum_{k\geq 1} \Phi_k^{(1)}\cos k\omega z+\Phi_k^{(2)}\sin k\omega z$$

is absolutely convergent. If $2\alpha_j\not\equiv 0$, $\alpha_j\pm\alpha_k\not\equiv 0 \pmod{\omega}$, $j\neq k$, $j, k=1, 2, \cdots, n$; and if either (i) Φ is symmetric; or (ii) $\Phi=(\Phi_{jk})$, $\Phi_{jk}(-z)=(-1)^{k+j}\Phi_{jk}(z)$, where each Φ_{jk} is a matrix; or (iii) if $\rho_{2k-1}^{(1)}=k\omega$, $\rho_{2k}^{(1)}=-k\omega$ are defined for those values of k for which $\Phi_k^{(1)}$ is non-zero and similarly for $\rho_{2k-1}^{(2)}, \rho_{2k}^{(2)}$, and if $\sum n_k^{(1)}\rho_k^{(1)}+n_k^{(2)}\rho_k^{(2)}=0$ (each $n_k^{(1)}$, $n_k^{(2)}=0$ or 1) implies $\sum n_k^{(2)}\equiv 0 \pmod 2$; then the solutions of (3) are bounded in $-\infty<z<\infty$, $0\leq|\lambda|\leq\lambda_0$, $\lambda_0>0$. Conditions (i) and (ii) have been previously discussed by L. Cesari [Atti. Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **11** (1941) 633–695; MR **8**, 208] and the reviewer [Rivista Mat. Univ. Parma **5** (1954), 137–167; Illinois J. Math. **1** (1957), 98–104; MR **17**, 36; **19**, 276] and can actually be treated without successive approximations using some results of A. Liapunov (see, e. g., V. A. Yakubovič [Prikl. Mat. Meh. **21** (1957), 707–713; MR **21** #158] and another proof of case (ii) by the reviewer [Arch. Rational Mech. Anal. **2** (1958/59), 429–434; MR **21** #1417]). However, case (iii) is new and seems to be difficult to obtain without successive approximation. *J. K. Hale* (Baltimore, Md.)

3625:

Golomb, Michael. Solution of certain nonautonomous differential systems by series of exponential functions. Illinois J. Math. **3** (1959), 45–65.

Let

$$\frac{dy}{dx} = f(y)+\sum_{k=1}^{\infty} g_k \exp(i\omega_k x)$$

be a differential equation for the n-dimensional vector function $y(x)$. The g_k are constant vectors, $\sum_{k=1}^{\infty}\|g_k\|<\infty$, and the ω_k are real numbers. Assume that $f(a)=0$ and that the components of $f(y)$ are holomorphic at $y=a$. Generalizing a result of Wasow [cited in review above], the author shows that this differential equation possesses, under fairly general conditions, a particular solution of the form

$$y = y^*(x) = a+\sum_{r\geq 1} a_r \exp(i\mu_r x),$$

where the a_r are constants, $\sum_{r\geq 1}\|a_r\|<\infty$, and the μ_r are finite linear combinations, with non-negative coefficients, of the ω_k. The most important condition concerns the eigenvalues λ_j of the Jacobian matrix A of $f(y)$ at $y=a$: the λ_j must be uniformly bounded away from the numbers $i\mu_k$. If one of the λ_j is a pure imaginary, $i\nu_0$, and one of the μ_k is a multiple of ν_0, then other particular solutions with certain expansions in terms of exponential functions exist that may be regarded as a generalization of subharmonics to the almost-periodic case. If all $\operatorname{Re}\lambda_j\leq 0$, all solutions $y(x; x_0, y_0)$ with initial values (x_0, y_0) sufficiently close to $(0, y^*(0))$ possess, for $0\leq x$, convergent series expansions in terms of certain exponential functions. These expansions are used to show that $y^*(x)$ is stable, and asymptotically stable if all $\operatorname{Re}\lambda_j<0$. Analogous results are proved for the more general equation $y'=g(x, y)$, if the right member possesses a convergent series expansion in powers of the components of $y-a$ with coefficients that can be developed in series in terms of the $\exp(i\omega_k x)$. *W. Wasow* (Madison, Wis.)

3626:

Opial, Z. Sur une inégalité de C. de la Vallée Poussin dans la théorie de l'équation différentielle linéaire du second ordre. Ann. Polon. Math. **6** (1959/60), 87–91.

The author considers the differential equation

$$(1)\qquad x''+g(t)x'+f(t)x = 0,$$

where $g(t)$ and $f(t)$ are real and continuous for all real t. Let $x(t)\not\equiv 0$ be a solution with successive zeros at $x(0)=0$ and $x(h)=0$, $h>0$. By improving a calculation of C. de la Vallée Poussin the author establishes the inequality $\pi^2\leq 4mh+kh^2$, where $2m=\max_{0\leq t\leq h}|g(t)|$ and $k=\max_{0\leq t\leq h}|f(t)|$. The equality sign is possible only when $m=0$ and thus this result is the best possible of its kind. *L. Markus* (Minneapolis, Minn.)

3627:

Opial, Z. Sur une critère d'oscillation des intégrales de l'équation différentielle $(Q(t)x')'+f(t)x=0$. Ann. Polon. Math. **6** (1959/60), 99–104.

Consider

$$(1)\qquad (Q(t)x')'+f(t)x = 0,$$

where $Q(t)>0$ and $f(t)$ are real continuous functions on $0\leq t<\infty$. Theorem 1: If $\int_0^\infty ds/Q(s)=+\infty$ and if there exists a positive function $\omega(t)\in C'$ such that

$$(2)\qquad \lim_{t\to\infty}\int_0^t \omega(s)\left[f(s)-\tfrac{1}{4}Q(s)\left(\frac{\omega'(s)}{\omega(s)}\right)^2\right]ds = +\infty,$$

then equation (1) is oscillatory at $+\infty$. For if (1) is not oscillatory there is a solution $x(t)$ for which $\limsup_{t\to\infty} x(t)=+\infty$. The corresponding solution $u(t)=Q(t)x'(t)/x(t)$ of the Riccati equation then exists for all

large t and this yields a contradiction to the hypothesis (2). Corollary: If $Q(t)\equiv 1$ and

$$\lim_{t\to\infty}\int_0^t s\left[f(s)-\frac{1}{4s^2}\right]ds=\infty$$

then equation (1) is oscillatory at $+\infty$. The constant $\frac{1}{4}$ is best possible in this case. Theorem 2: If the equation

$$(3)\qquad x''+f(t)x=0$$

is not oscillatory at $+\infty$, then there exists a solution $x(t)$ such that measure $\{x(t)<k\}<\infty$ for each finite k.

L. Markus (Minneapolis, Minn.)

3628:

Kondrat'ev, V. A. Oscillation of solutions of linear equations of third and fourth order. Trudy Moskov. Mat. Obšč. 8 (1959), 259–281. (Russian)

This is a complete treatment of the results summarized in a note [Dokl. Akad. Nauk SSSR **118** (1958), 22–24; MR **20** #146. Additional references: Davidaglan, Ann. Ec. Normale **22** (1905); Mikusiński, Ann. Polon. Math. **1** (1955), 207–221; MR **19**, 141; Švec, Czechoslovak Math. J. **5** (**80**) (1955), 29–60; MR **17**, 612; Mammana, Math. Z. **33** (1931), 186–231; Sobol, Dokl. Akad. Nauk SSSR **61** (1948), 219–222; MR **10**, 40].

S. Lefschetz (Princeton, N.J.)

3629:

Jones, John, Jr. On nonlinear second order differential equations. Proc. Amer. Math. Soc. **9** (1958), 586–588; errata, **10** (1959), 1000.

In this note the author gives sufficient conditions that the differential equation $y''+\sum_{i=1}^n f_i(x)y^{2i-1}=0$ has no solution with arbitrarily large zeros. The methods and results are similar to those of F. V. Atkinson [Pacific J. Math. **5** (1955), 643–647; MR **17**, 264]. {The author has informed the reviewer that more conditions are given in this paper than are necessary.}

Choy-tak Taam (Washington, D.C.)

3630:

Yakubovič, V. A. Oscillatory properties of the solutions of standard linear sets of simultaneous differential equations. Dokl. Akad. Nauk SSSR **124** (1959), 533–536. (Russian)

The results announced in this paper contribute to several aspects of oscillation theory, and the paper goes far towards putting the theory in a simple but general form. The primary interpretation of "oscillation" is here in terms of the behaviour of an angle $\phi=\operatorname{Arg} X$, associated with a varying symplectic matrix X; other interpretations, concerning zeros, are consequential.

Let $\mathfrak{G}$ be the symplectic group of $2k$-by-$2k$ matrices X, characterised by $X^*JX=J$, where $J=\begin{pmatrix}0 & I_k\\ -I_k & 0\end{pmatrix}$, and I_k is the kth order unit matrix. The first section of the paper considers varying definitions of $\operatorname{Arg} X$, $X\in\mathfrak{G}$. The original definition of I. M. Gel'fand and V. B. Lidskiĭ [Uspehi Mat. Nauk (N.S.) **10** (1955), no. 1 (63), 3–40; MR **17**, 482] depended on the representation $X=SU$, with S positive-definite, U orthogonal. The present author gives five further definitions, depending explicitly on X.

For example, if

$$X=\begin{pmatrix}U_1 & U_2\\ V_1 & V_2\end{pmatrix},$$

then $\operatorname{Arg}_1 X=\operatorname{Arg}\det(U_1+iU_2)$. The resulting six definitions do not give the same value, but are "equivalent" in the sense that the difference between any two determinations of $\operatorname{Arg} X$ is uniformly bounded as X varies over any continuous arc in $\mathfrak{G}$.

Let now $X(t)$ be the solution of $dX/dt=JH(t)X$, $X(0)=I_{2k}$, where $H(t)$ is real, symmetric and Lebesgue integrable. The equation is "oscillating" if $\operatorname{Arg} X(t)$ is unbounded as $t\to+\infty$, and "of positive type" if, for some determination, $\operatorname{Arg} X(t)$ is monotonic increasing. Conditions for these situations concern the positive-definiteness of certain sub-matrices of $H(t)$ [cf. V. B. Lidskiĭ, Dokl. Akad. Nauk SSSR **102** (1955), 877–880; MR **17**, 483].

The author next considers two special cases in which the oscillatory property has been formulated in other ways. One of these is the equation $(d/dt)[R(t)y]+P(t)y=0$, where y is a k-vector and R and P symmetric matrices, with $R>0$; this may be said to be oscillating if for any t_0 there exist $t_2>t_1>t_0$, and $y(t)\not\equiv 0$, such that $y(t_1)=y(t_2)=0$. In the case of a scalar equation

$$\sum_{j=0}^{k}(-1)^j[\phi_j(t)\eta^{(k-j)}]^{(k-j)}=0,\quad \phi_0(t)>0,$$

one requires that some $\phi\not\equiv 0$ have a k-fold zero at t_1, t_2. These requirements are said to agree with the definition of "oscillating" given in the general case.

The paper concludes with applications to the boundary problem $dx/dt=J[H_0(t)+\lambda H_1(t)]x$, $x(\tau)=Sx(0)$, where x is a $2k$-vector and $S\in\mathfrak{G}$. Defining $X(t)=X(t,\lambda)$ as before, one result is that there are infinitely many eigen-values $\lambda_n\to\infty$ if and only if $\lim_{\lambda\to\infty}\operatorname{Arg} X(\tau,\lambda)=\infty$; a similar result holds concerning the possibility $\lambda_n\to-\infty$.

F. V. Atkinson (Canberra)

3631:

Yakubovič, V. A. Oscillation and non-oscillation conditions for canonic linear sets of simultaneous differential equations. Dokl. Akad. Nauk SSSR **124** (1959), 994–997. (Russian)

The author considers linear systems of the canonic Hamiltonian form (1) $dx/dt=JH(t)x$, where x is a $2k$-vector, J, H are $2k\times 2k$ matrices, $J=(0, I_k;\ -I_k, 0)$, I_k the unit matrix of order k, and $H(t)$ a real symmetric matrix whose elements are functions of t, $0\le t\le+\infty$, L-integrable in every finite interval. Any two solutions x_1, x_2 of system (1) have the property that $(Jx_1, x_2)=$ constant. Given any k real independent solutions of (1), if we write the $2k\times k$ matrix $[x_1,\cdots,x_k]$ in the form col (U, V), where U, V are $k\times k$ matrices, then $\det[U(t)-iV(t)]\neq 0$ for all $t\ge 0$, as the author has proved in a previous paper [reviewed above]. If the $2k\times 2k$ matrix $X(t)$ denotes a fundamental system of solutions of (1), and $A=U(0)$, $B=V(0)$, $(M,N)=X\cdot(A,B)$, then $\operatorname{Arg} X(t)=\operatorname{Arg}\det(M-iN)$ is assumed as a definition of $\operatorname{Arg} X(t)$, as in the previous paper. System (1) is said to present oscillations [no oscillations] if $\operatorname{Arg} X(t)$ is unbounded [bounded] as $t\to+\infty$. Conditions are given to assure either character of system (1). For instance, let $h_1(t)\le\cdots\le h_{2k}(t)$ be the $2k$ characteristic roots of the symmetric matrix $H(t)$, and $u(t)=\int_0^t(h_1+\cdots+h_k)dt$, $v(t)=\int_0^t(h_{k+1}+\cdots+h_{2k})dt$. I: If either lim sup $u(t)=+\infty$, or lim inf $v(t)=-\infty$ as $t\to+\infty$, then (1) presents oscillations; if both $u(t)$, $v(t)$ are bounded as $t\to+\infty$, then (1) presents no oscillations.

L. Cesari (Baltimore, Md.)

3632:

Zubov, V. I. Some problems in stability of motion. Mat. Sb. (N.S.) **48 (90)** (1959), 149–190. (Russian)

Three problems are treated at length in this basic paper: I, the analytic representation of a solution of a system of partial (or ordinary) differential equations near a singular point; II, the stability of an equilibrium point of a dynamical system and the qualitative behavior of nearby trajectories; III, the stability of the trivial solution of a system of differential equations in a number of doubtful cases where certain eigenvalues are zero or pure imaginary. Some of these results have been announced previously by the author [Dokl. Akad. Nauk SSSR **109** (1956), 1095–1097; **110** (1956), 169–171; **114** (1957), 942–944; MR **18**, 578, 396; **19**, 857].

I. The author considers the system

$$(1)\quad \frac{\partial z_j}{\partial t}+\sum_{s=1}^{n}\frac{\partial z_j}{\partial x_s}\left(\sum_{i=1}^{n}p_{si}(t)x_i+X_s(t,x,z)\right)=\sum_{i=1}^{k}q_{ji}(t)z_i+\sum_{i=1}^{n}r_{ji}(t)x_i+Z_j(t,x,z),$$

where $j=1,\cdots,k$, $x=(x_1,\cdots,x_n)$, and $z=(z_1,\cdots,z_k)$. Theorems 1–3 have to do with the solutions of (1). Theorem 1 is typical. Let $\mu_1,\cdots,\mu_n$ be the Lyapunov numbers associated with the system (2) $dx/dt=P(t)x$, where $P(t)=((p_{si}(t)))$; and $\mu_{n+1},\cdots,\mu_{n+k}$ those associated with the system (3) $dz/dt=Q(t)z$, where $Q(t)=((q_{ji}(t)))$. Suppose $\mu_i>0$ for $i\leq n$, $\mu_i=\mu_{n+i}$ for $i\leq l$, and (2) and (3) are Lyapunov regular. Then (1) has a family of solution (depending upon l arbitrary constants) of the form $z_j=\sum_{m=1}^{\infty}z_j^{(m)}(t,c,x)$, $j=1,\cdots,k$, where $c=(c_1,\cdots,c_l)$. This series converges for $|c_s|<c_0$, a positive constant, and $|x_s|<x_0(t)$, a positive function converging to 0 sufficiently fast as $t\to\infty$. The proofs involve the classical Lyapunov expansion theorems for the solution of a system of ordinary differential equations [*Problème général de la stabilité du mouvement*, Princeton Univ. Press, Princeton, N.J., 1947; MR **9**, 34]. These results are then applied in generalizing some theorems of Briot and Bouquet [J. École Polytech. **21** (1856), 85–254], Poincaré [ibid. **28** (1878), 13–26], Picard [*Traité d'analyse*, v. 3, Paris, Gauthier-Villars, 1896], and Horn [*Gewöhnliche Differentialgleichungen beliebiger Ordnung*, Göschen, Leipzig, 1934]. Of the six theorems the author proves, the following is typical. Theorem 4: Consider the system

$$(4)\quad z^N dy_s/dz=\sum_{i=1}^{n}p_{si}(z)y_i+p_s(z)z+Y_s(z,y),$$

where $y=(y_1,\cdots,y_n)$ and Y_s is a convergent series of the form

$$\sum P_s^{(m,m_1,\cdots,m_n)}(z)z^m y_1^{m_1}\cdots y_n^{m_n}$$

without linear terms. Let $\mu_1,\cdots,\mu_n$ be the Lyapunov numbers associated with the system (5) $dy/dt=-P(e^{-t})y$, where $P(t)=((P_{si}(t)))$. If $N=1$, if (5) is Lyapunov regular, and if $\mu_i>0$ for $i\leq l$, then (4) has a family of solutions

$$y_s=\sum K_s^{(m,m_1,\cdots,m_l)}(z)(z^{m+\sum_{i=1}^{l}m_i\mu_i})c_1^{m_1}\cdots c_l^{m_l},$$

$s=1,\cdots,n$, convergent in a suitable c,z region, where $K_s\cdot z^{\alpha}\to 0$ as $z\to 0$ for any constant α.

II. Using theorems of Krasovskiĭ [Akad. Nauk SSSR. Prikl. Mat. Meh. **18** (1954), 513–532; MR **16**, 473] and Erugin [ibid. **14** (1950), 459–512; MR **12**, 412] on the inverse theorems of Lyapunov, the author proves four theorems on the stability of the trivial solution for system (6) $dx/dt=f(x)$, $x=(x_1,\cdots,x_n)$, where $f(0)=0$. Typical is theorem 11: The trivial solution of (6) is Lyapunov asymptotically stable if and only if (1) there is no trajectory (other than $x=0$) of (6) for which 0 is an α limit point and (2) there is a sufficiently small neighborhood of the origin containing no complete trajectory of (6). The author then turns to (7) $dx/dt=X^{(\mu)}(x)+\sum_{m>\mu}X^{(m)}(t,x)$, where $X^{(\mu)}$ is a vector form of degree $\mu\geq 1$ in the n-vector x, and the $X^{(m)}$s are vector forms of degree m in x with coefficients which are real, bounded and continuous functions of t for $t\geq 0$. (7) is said to be asymptotically stable if the trivial solution is asymptotically stable regardless of the $X^{(m)}$, $m>\mu$. A number of theorems (14–17) and corollaries are proved giving necessary and sufficient conditions for the asymptotic stability of the trivial solution of (8) $dx/dt=X^{(\mu)}(x)$, and hence for the asymptotic stability of (7). For further reference see the article of Massera [Ann. of Math. (2) **64** (1956), 182–205; MR **18**, 42]. Now let $z(t)$ be a positive solution of (9) $dz/dt=-az^{\mu}$, $a=\pm 1$. An integral curve, $x=x(t)$, of (7) is said to have a definite direction, b, if $\lim_{at\to+\infty}x(t)/z(at)=b$, where $b=(b_1,\cdots,b_n)$ and the b's are real constants. The author proves theorems 19 and 20 giving necessary and sufficient conditions that (7) be asymptotically stable and have a finite number of direction numbers and that each solution of (7) have a direction. Theorem 18 gives a series expansion of an O-curve of (7).

III. Consider the system (10) $dx/dt=X(x,y)$, $dy_j/dt=Py+Y(x,y)$, $x=(x_1,\cdots,x_k)$, $y=(y_1,\cdots,y_k)$, P a constant matrix, X and Y convergent power series for small enough x, y with no linear or constant terms. Theorems 21, 22 and 23 give information about the structure of integral curves near the trivial solution of (10) and conditions for the stability of that solution. The author concludes with two theorems giving conditions for the Lyapunov stability of the trivial solution of

$$(11)\quad \begin{aligned} dx_s/dt&=-\lambda_s y_s+X_s(x,y,z),\\ dy_s/dt&=\lambda_s x_s+Y_s(x,y,z),\\ dz/dt&=Rz+Z(x,y,z),\end{aligned}$$

where $s=1,\cdots,k$, $x=(x_1,\cdots,x_k)$, $y=(y_1,\cdots,y_k)$, and $z=(z_1,\cdots,z_n)$.

The statement and proofs of many of the theorems of this paper are also to be found in Zubov's book [*Metody A. M. Lyapunova i ih primenenie*, Izdat. Leningrad. Univ., Moscow, 1957; MR **19**, 275] and in a report by S. Lefschetz [RIAS Tech. Rep. **58-2**, Baltimore].

C. S. Coleman (Claremont, Calif.)

3633:

Gröbner, Wolfgang. Le soluzioni generali del problema degli n corpi rappresentate mediante serie di Lie. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 11–15.

Consider the differential system $\dot{x}_k=u_k$, $\dot{u}_k=-\partial U/\partial x_k$ where $U=U(x_1,\cdots,x_n)$. By means of the differential operator

$$D=\sum_{k=1}^{n}\left(u_k\frac{\partial}{\partial x_k}-\frac{\partial U}{\partial x_k}\frac{\partial}{\partial u_k}\right)$$

one may define the Lie series

$$F(X_1,\cdots,U_n)=\sum_{\nu=0}^{\infty}\frac{1}{\nu!}t^{\nu}D^{\nu}F(x_1,\cdots,u_n),$$

where F is an arbitrary analytic function of the $2n$ variables $X_k(t)$, $U_k(t)$ which solve the above differential system with initial values x_k, u_k. The author considers the case of the n-body problem where U represents the gravitational interaction of the n points. He states that the Lie series converges absolutely in the complex t-plane until a collision point is reached on the circumference $|t|=T$. The ten algebraic integrals of motion are derived from the series development and a result on inversion of Lie series is announced. *M. Schiffer* (Stanford, Calif.)

3634:

Razumova, E. F. The set of branching points when there is no uniqueness to one side. Dokl. Akad. Nauk SSSR **125** (1959) 978–981. (Russian)

The author considers the dynamical system governed by the equations

$$dx/dt = X(x, y), \quad dy/dt = Y(x, y),$$

where (X, Y) is a continuous vector field without singular points on a simply connected plane region σ having a piece-wise smooth boundary. A point of one-sided non-uniqueness in $\bar{\sigma}$ is a point from which more than one integral curve issues in the direction of increasing (or decreasing) time. The set of all integral curves issuing from such a point c in the direction of the vector field at that point is called a funnel with vertex c. The author states ten theorems (no proofs) which give information concerning the possible geometric structure of the funnels, vertices, and trajectories with one-sided non-uniqueness points. *E. A. Coddington* (Los Angeles, Calif.)

3635:

Andreev, A. F. On the first problem of distinguishing in Frommer's theory. Vestnik Leningrad. Univ. **14** (1959), no. 7, 18–25. (Russian. English summary)

The author considers a problem concerning the number of solutions of the differential equation

$$(1) \qquad \frac{dy}{dx} = \frac{Q(x, y)+q(x, y)}{P(x, y)+p(x, y)}$$

—in polar coordinates

$$(1') \qquad r\frac{d\varphi}{dr} = \frac{F(\varphi)+f(r, \varphi)}{G(\varphi)+g(r, \varphi)}$$

—which tend to a critical point tangent to some fixed direction. In particular he assumes that: (a) P and Q are homogeneous polynomials of integral degree $n \geq 1$; (b) p and q are continuous in some neighborhood N of the origin, satisfy some condition ensuring the uniqueness of the solution of the Cauchy problem in N, and are such that $r^{-n}p(x, y)$ and $r^{-n}q(x, y) \to 0$ as $r \to 0$; (c) f and g satisfy conditions similar to those in (b) above except that $r^{-1}f$ and $r^{-1}g \to 0$ as $r \to 0$ and $f(0, \varphi)=g(0, \varphi)=0$ by definition; (d) $F(0)=0$; (e) in the expansions $F(\varphi)=C\varphi^k+\cdots$, $G(\varphi)=G_0+G_1\varphi+\cdots$, k is odd and $CG_0<0$. It is known that under these conditions at least one solution of (1′), $\varphi=\varphi_1(r)$, $\to 0$ as $r \to 0$. The question of whether this is the only solution with this property or whether there are infinitely many is known as the first problem of distinguishing in Frommer's theory [Frommer, Math. Ann. **99** (1928), 222–272]. Lonn [Math. Z. **44** (1938), 507–530] proved that if $k=1$ then for uniqueness it is sufficient that $r^{-n}p(r \cos \varphi$, $r \sin \varphi)$ and $r^{-n}q(r \cos \varphi,\ r \sin \varphi)$ satisfy some Lipschitz condition in φ with constant $L(r)$ which tends to 0 with r. Haimov [Uč. Zap. Stalinabad. Gos. Univ. **1**, **2** (1952); Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **21** (1957), 113–122; MR **20** #2496] proved that if p and q are analytic in some neighborhood of the origin, then there is uniqueness whether or not $k=1$.

In this paper the author proves that for uniqueness to hold under conditions (a)–(d) above it is sufficient that it hold for the equation

$$(2) \qquad r\frac{d\varphi}{dr} = \frac{F(\varphi_1(r))}{G(\varphi_1(r))+g(r, \varphi_1(r))}+\frac{f(r, \varphi)}{G(\varphi)+g(r, \varphi)}.$$

Two corollaries are proved showing that the author's results include those of Lonn and of Haimov.

C. S. Coleman (Claremont, Calif.)

3636:

Coleman, Courtney. A certain class of integral curves in 3-space. Ann. of Math. (2) **69** (1959), 678–685.

The author considers the system (*) $dx/dt=f(x)$, where f is a 3-dimensional vector whose components are homogeneous polynomials of degree $m \geq 1$ in the components of x.

This homogeneity implies that the homothetic of an integral curve is also an integral curve. Things can then be reduced to the sphere S^2 and one thus gets a classification of the integral surfaces of (*) and a partial classification of the integral curves on those surfaces.

M. M. Peixoto (Rio de Janeiro)

3637:

Agranovič, M. S. General solutions of differential-difference equations with constant coefficients. Dokl. Akad. Nauk SSSR **123** (1958), 9–12. (Russian)

The author considers the equation

$$\sum_{k=0}^{l} P_k(\partial/\partial x)u(x-h_k) = f(x),$$

where $x=(x_1, \cdots, x_n)$ is a point of an n-dimensional real space R; $(\partial/\partial x)=(\partial/\partial x_1,\ \partial/\partial x_2, \cdots, \partial/\partial x_n)$; the P_k are polynomials with constant complex coefficients; and the h_k are fixed points of R, where $h_0=0$ and $h_j \neq h_k$ for $j \neq k$. He obtains a formal solution in terms of "generalized functions" described by Gel′fand and Šilov [*Obobščennye funkcii i deĭstviya nad nimi*; *Prostranstva osnovnyh i obobščennyh funkciĭ*; Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958; MR **20** #4182; **21** #5142a] and related to the "distributions" of L. Schwartz [*Théorie des distributions*, Hermann, Paris, 1950 and 1951; MR **12**, 31, 833].

T. N. E. Greville (Kensington, Md.)

3638:

Massera, J. L.; and Schäffer, J. J. Linear differential equations and functional analysis. III. Lyapunov's second method in the case of conditional stability. Ann. of Math. (2) **69** (1959), 535–574.

In this third part of their work on linear differential equations

$$(1) \qquad \dot{x}+A(t)x = 0$$

in a Banach space X [for parts I, II see same Ann. (2) **67** (1958), 517–573; **69** (1959), 88–104; MR **20** #3466; **21** #756], the authors take up the (very considerable) problem of extending Lyapunov's second method to the case of

conditional, i.e., non-uniform, simple and asymptotic stability. They are able to prove both "direct" and "converse" theorems and thereby establish the equivalence between the existence of suitably generalized Lyapunov functions with certain properties and the asymptotic behavior of the bounded and unbounded solutions of (1). These theorems they combine with previous ones of part I to obtain a complete characterization, in terms of Lyapunov functions, of the existence of at least one bounded solution of

$$(2) \qquad \dot{x}+A(t)x = f(t)$$

with f from a certain function space. The importance of these results, as indeed of all results in Lyapunov's method, lies in the fact that they provide testable criteria in the sense that all properties of the Lyapunov functions involve solely the (homogeneous) differential equation and require therefore no knowledge of the solutions at all.

Only a bare outline of the wealth of theorems proved in this paper can be given here. Recall that, throughout, A is a continuous endomorphism of X, defined on $J=[0, \infty)$ and uniformly Bochner integrable on every finite subinterval of J; M denotes the space of such endomorphisms (or functions) for which $\int_t^{t+1} \|A(t)\|dt$ is bounded on J.

The authors first define the concepts of a Lyapunov function $V(x, t)$ on $X\times J$ and of a total derivative $V'(x, t)$ in such a way that the following crucial implication is true under sufficiently general assumptions: If, for a continuous function $a(x, t)$ on $X\times J$, $V'\geqq a$ holds "almost everywhere" (in a sense made precise in the paper), then for any $t\geqq t_0\geqq 0$

$$V[x(t), t]-V[x(t_0), t_0] \geqq \int_{t_0}^{t} a[x(t), t]dt,$$

where $x(t)$ is a solution of (1). The terms positive definite, infinitely small upper bound are defined as usual. V' is called positive almost definite if V' is positive definite save on a set H such that either its projection on J is a Lebesgue null set or, if $\dim X<\infty$, H itself is such a set; and if V has certain regularity properties. The first two main theorems can then be stated, in part, as follows.

(I) If $A\in M$ and if there exists a V with infinitely small upper bound and one of its total derivatives V' is negative almost definite, then (i) the set X_0 of initial values of bounded solutions is closed and

$$(3) \qquad \|x(t)\| \leqq N\|x(t_0)\|e^{-\nu(t-t_0)}, \quad t\geqq t_0\geqq 0,$$

N, ν being positive constants; (ii) if X_1' is a subspace such that either $x\in X_1'$ implies $V(x, 0)\leqq 0$ $[V(-x, 0)\leqq 0]$ or $X_1'\cap X_0=\emptyset$ and $\dim X_1'<\infty$, then $x(0)\in X_1'$ implies

$$(4) \qquad \|x(t)\| \geqq N'\|x(t_0)\|e^{\nu'(t-t_0)}, \quad t\geqq t_0\geqq 0,$$

N', ν' being positive constants; (iii) if $x_0(0)\in X_0, x_1(0)\in X_1$ then (5) $\alpha[x_0(t), x_1(t)]\geqq\alpha_0$ for $t\geqq 0$, where α is the angular distance defined in part I. The most striking feature is that V is not required to be positive definite as it is in the classical theory. There $A\in M$ need not hold; here (I) is false if $A\in' M$.

(II) If X_0, $X_1=\complement X_0$ are closed, and if $x_0(t_0)\in X_0$ implies (3), $x_1(0)\in X_1$ implies (4), and $x_i(0)\in X_i$ imply (5); then there exist non-negative functions V_0, V_1 such that every total derivative of V_0-V_1 is negative almost definite.

Combined with results from part I these theorems yield the following.

(III) If the hypotheses of (I) hold and if either $\operatorname{codim} X_0<\infty$ or $X_1=\complement X_0$ is closed and $x\in X_1$ implies $V(x, 0)\leqq 0$ $[V(-x, 0)\leqq 0]$, then (2) has at least one bounded solution for every $f\in M$.

(IV) If $A\in M$ and X_0, $X_1=\complement X_0$ are closed and if the assertion of (III) holds, then the assertion of (II) holds.

The authors point out that these results are closely related to work of Krasovskiĭ [Mat. Sb. (N.S.) **40** (82) (1956), 57–64; MR **19**, 34].

In the case of simple conditional stability the authors prove theorems of analogous type. Roughly speaking, they are able to infer from the existence of non-negative, positively homogeneous Lyapunov functions that every bounded solution satisfies $\|x(t)\|\leqq N_0\|x(t_0)\|$, $t\geqq t_0\geqq 0$, and for the other solutions $\|x(t)\|\geqq N'\|x(t_0)\|$. Conversely, if this dichotomy holds, there exist such functions. By virtue of theorems of part I there follows a characterization of the bounded solutions of (2) similar to that in (III) and (IV).

Throughout the paper the authors give numerous examples to illustrate the importance of the various hypotheses in the theorems. The methods of proof are essentially classical.

H. A. Antosiewicz (Los Angeles, Calif.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 3542, 3601, 3611, 3685, 3708, 4034, 4035a–b, 4036.

3639:

★Tychonoff, A. N.; und Samarski, A. A. Differentialgleichungen der mathematischen Physik. Hochschulbücher für Mathematik, Bd. 39. VEB Deutscher Verlag der Wissenschaften, Berlin, 1959. 660 pp. DM 42.00.

This is a German translation of the 2nd edition of the Russian original [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953; MR **16**, 364]. The first edition of the original (1951) was reviewed in MR **15**, 430.

3640:

Bagrinovskiĭ, K. A.; and Godunov, S. K. Difference schemes for multidimensional problems. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 431–433. (Russian)

The article gives a method to construct a difference scheme for the solution of many-dimensional hyperbolic differential equations with constant coefficients of the form

$$\frac{\partial u_i}{\partial t} = \sum_{j=1}^{m}\sum_{k=1}^{n} a_{ij}{}^k\frac{\partial u_j}{\partial x_k}+\sum_{j=1}^{m} b_{ij}u_j \quad (i=1,\cdots,m)$$

and to decide upon its stability. It is constructed out of stable difference schemes for the n systems of one-dimensional differential equations

$$\frac{\partial u_i^{(k)}}{\partial t} = \sum_{j=1}^{m} a_{ij}{}^k\frac{\partial u_j^{(k)}}{\partial x_k}+\sum_{j=1}^{m} b_{ij}{}^k u_j^{(k)}$$

$$(i=1,\cdots,m;\quad k=1,\cdots,n),$$

where $b_{ij}{}^k$ are arbitrary numbers with $\sum_{k=1}^{n} b_{ij}{}^k=b_{ij}$.

The very concise style makes it difficult to follow the argument.

W. H. Muller (The Hague)

3641:

Yanenko, N. N. A difference method of solution in the case of the multidimensional equation of heat conduction. Dokl. Akad. Nauk SSSR **125** (1959), 1207–1210. (Russian)

Bagrinovskiĭ and Godunov [Dokl. Akad. Nauk SSSR **115** (1957), 431; MR **21** #3640] devised a "method of decomposition" for reducing many-dimensional explicit difference schemes approximating hyperbolic differential-equation systems to 2-dimensional ones. The author shows that the method of decomposition can be applied to prove the known stability and convergence of a family of implicit difference schemes for the heat equation in many dimensions. The problem is to solve

$$\frac{\partial u}{\partial t} = a^2 \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2}$$

for $0 \leq x_i \leq 1$, $0 \leq t \leq T$, with prescribed conditions for $t=0$ and for $x_i=0$, 1 $(i=1, \cdots, m)$. The difference scheme, which is familiar, employs centered second differences in the x_i-directions, and uses a weighted average of 2 time levels. The decomposition is too technical to reproduce.

G. E. Forsythe (Stanford, Calif.)

3642:

★Lepage, Th. H. Sur une classe d'équations non linéaires du second ordre. La théorie des équations aux dérivées partielles. Nancy, 9–15 Avril 1956, pp. 111–115. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXI. Centre National de la Recherche Scientifique, Paris, 1956. 187 pp. 1500 francs.

Let $X=(x_{rs})_{1\leq r\leq n, 1\leq s\leq n}$, where $x_{rs}=\partial^2 v/\partial x_r \partial x_s$ and $n \geq 2$. The author considers equations of the form $L(X)=0$, L being a linear combination of the minors of X. He gives a brief description of certain algebraic results about $L(X)$ involving exterior forms and the symplectic group, most of which seem to be from earlier uncited papers [e.g., *Premier colloque sur les équations aux derivées partielles*, Louvain 1953, pp. 79–104, Thone, Liège, Masson, Paris, 1954; MR **16**, 1028], and in a final section, hereby quoted in its entirety, hints at analytic applications: "Un résultat typique est le suivant: toute solution de l'équation $rt-s^2-1=0$ est analytique; toute solution entière est un polynôme du second degré. Même résultat pour l'équation $L_4(X)-1=0$." *E. R. Kolchin* (New York, N.Y.)

3643:

Hersch, Joseph. Une interprétation du principe de Thomson et son analogue pour la fréquence fondamentale d'une membrane. Application. C. R. Acad. Sci. Paris **248** (1959), 2060–2062.

The author explores the idea of reducing the value of a minimum in a minimum principle by admitting functions with discontinuities along certain curves. This gives lower bounds for the torsional rigidities of certain domains. The author also finds that for certain domains depending on a parameter, the torsional rigidity is a convex functional of this parameter.

In the case of a fixed vibrating membrane the author finds that the lowest eigenvalue is no smaller than the lowest eigenvalue of any of the membranes obtained by cutting the original membrane and regarding the cuts as free edges. Cutting the membrane into infinitesimal strips leads to a lower bound for the lowest eigenvalue in terms of one-dimensional problems, which generalizes results of L. E. Payne and the reviewer [J. Soc. Indust. Appl. Math. **5** (1957), 171–182; MR **19**, 1110].

H. F. Weinberger (College Park, Md.)

3644:

★Peetre, Jaak. Estimates for the number of nodal domains. Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18–23 août 1957, pp. 198–201. Mercators Tryckeri, Helsinki, 1958. 209 pp. (1 plate)

Let Ω be a relatively compact domain in a smooth two-dimensional manifold $\mathfrak{M}$. Let N be the number of nodal domains of the nth eigenfunction of $\Delta u - \lambda u = 0$ in Ω, $u=0$ on $\mathfrak{b}\Omega$, where Δ is the Laplace-Beltrami operator and $\mathfrak{b}\Omega$ is the boundary of Ω. Under the assumption that $\mathfrak{M}$ is homeomorphic to a domain in the Euclidean plane, the author proves that $N<n$ when n is sufficiently large. The theorem had previously been proved by Å. Pleijel [Comm. Pure Appl. Math. **9** (1956), 543–550; MR **18**, 315] when $\mathfrak{M}$ is the Euclidean plane and by the author [Math. Scand. **5** (1957), 15–20; MR **19**, 1180] when $\mathfrak{M}$ is a Riemannian manifold with certain properties. That $N \leq n$ for all n is the statement of Courant's nodal domain theorem [Nachr. Gesell. Wiss. Göttingen, Math.-Phys. Kl. **1923**, 81–84].

M. Schechter (New York, N.Y.)

3645:

Naimark, M. A. Spectral analysis of non-self-adjoint operators. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 6 (72), 183–202. (Russian)

This survey of results obtained on the spectral theory of non-self-adjoint operators is primarily concerned with ordinary and partial differential operators and their eigenfunction expansions. It is divided into three sections, the first concerned with regular differential operators, i.e. those for which the resolvent is compact, the second with singular differential operators, and the third with general results on spectral theory for operators in Banach spaces and non-self-adjoint operators in Hilbert space. Section 1 begins with an account of the results of Birkhoff on eigenfunction expansions for ordinary differential operators which are not symmetric, and their extensions due to Tamarkin, Langer, Birkhoff and Langer, and M. Stone. Turning to partial differential operators, the writer summarizes Carleman's results on the eigenvalue distribution of second-order linear elliptic operators [Ber. Verh. Sächs. Akad. Wiss. Leipzig, **88** (1936), 119–134]. The section concludes with a detailed account of results on completeness of eigenfunctions for elliptic boundary-value problems (non-self-adjoint) due to M. V. Keldyš [Dokl. Akad. Nauk SSSR **77** (1951), 11–14; MR **12**, 835] and the reviewer [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 433–439; MR **14**, 984]. Section 2 gives a summary of results due to the author [Trudy Moskov. Mat. Obšč. **3** (1954), 181–270; MR **15**, 959] on the spectral theory of the ordinary differential operator $(-D^2+p(x)I)$ on the half-axis, as well as related results of B. Levin and J. T. Schwartz. In addition, a brief description is given of results on the Schroedinger operator with complex potential due to Gel'fand [Uspehi Mat. Nauk (N.S.) **7** (1952), no. 6 (52), 183–184; MR **14**, 1091] and unpublished results of R. M. Martirosyan. Section 3 begins with a detailed summary of the results of Lifschitz [Mat. Sb. (N.S.) **34 (76)** (1954), 145–199; Amer. Math. Soc. Transl. (2) **5** (1957), 67–114; MR **16**, 48; **18**, 748] on the triangular form for operators whose

skew-Hermitian part is of finite trace class, as well as related results of Sahnovič and of Mukminov [Dokl. Akad. Nauk SSSR **99** (1954), 499–502; MR **16**, 830]. The section closes with a brief summary of some results in theory of spectral operators in Banach spaces due to Dunford and his school, as well as related results of M. K. Fage [ibid. **58** (1947), 1609–1612; MR **9**, 290] and F. Wolf [Proc. Internat. Congress Math. 1954, Amsterdam, vol. II, pp. 186–187, Noordhoff, Groningen, 1956].

F. Browder (New Haven, Conn.)

3646:

Berezanskiĭ, Yu. M. Eigenfunction expansions of self-adjoint operators. Mat. Sb. N.S. **43** (85) (1957), 75–126. (Russian)

The author deals with operators in $L^2(E^n)$ and strongly relies on the theory of finite distributions. Let $\omega_x(\xi)$ be the characteristic function multiplied by $\prod_1^n \operatorname{sgn} x_i$ of the parallelepiped whose opposite corners are at 0 and x, and $D=\partial^n/\partial x_1\cdots\partial x_n$. The first theorem is as follows. If A is hermitian and E_λ its spectral measure, then there exists a non-decreasing scalar function ρ such that $\Theta(x,y,\lambda)=(E_\lambda\omega_y,\omega_x)$ and $\Psi(x,y,\lambda)$ (its ρ-derivative) are positive definite kernels and

$$(E_\Delta f,g)=\int_\Delta d_\lambda\left\{\iiint\Theta(x,y,\lambda)Df(y)\overline{Dg(x)}dxdy\right\}=$$
$$\int_\Delta\left\{\iiint\Psi(x,y,\lambda)Df(y)\overline{Dg(x)}dxdy\right\}d\rho(\lambda),$$

where f,g are arbitrary n-times differentiable functions of compact support. For all $\lambda\in E^1$ the kernel Θ is continuous in x, y. Ψ is summable with its square in every compact subset of E^m. Every such integral will be bounded in λ. For almost all (with respect to ρ) λ and almost all y, $D_x\Psi$ (analogously $\overline{D_y\Psi}$) is a generalized eigenvector of A in the following sense. There exists a sequence u_n such that both $\|u_n\|$ and $\|Au_n\|$ are finite, $\langle u_n, Au_n\rangle$ is dense in the graph of A, all n-times differentiable functions, and $\int\psi\overline{D(A-\lambda I)u_n}dx=0$. The proof is the obvious formal procedure made precise. The functions ρ, Θ, Ψ are unique. A further result is the following: There exists a τ positive in E^n and tending to zero as $x\to\infty$, and a finite or infinite sequence of functions $\Psi_\alpha(x,\lambda)$ for which $D_x\Psi_\alpha$ is a generalized eigenfunction of A such that $\Psi(x,y,\lambda)=\sum_1^{N(\lambda)}\Psi_\alpha(x,\lambda)\overline{\Psi_\alpha(y,\lambda)}$. The convergence of this series if infinite is in the sense of $L^2(\tau(x)\tau(y)dxdy)$. To separable hermitian operators is applicable the theorem: If $E_n=E_{n'}\oplus E_{n''}$, $x=\langle x',x''\rangle$, $f'\in L^2(E^{n'})$, $f''\in L^2(E^{n''})$ and $A(f'f'')=A'(f')f''+f'A''(f'')$ the corresponding kernels, then

$$\Theta(x,y,\lambda)=\Theta(\langle x',x''\rangle,\langle y',y''\rangle,\lambda)$$
$$=\int_{-\infty}^{\infty}\Theta'(x',y',\lambda-\mu)d_\mu\Theta''(x'',y'',\mu).$$

The remainder of the paper deals with applications to partial differential equations, especially to those of elliptic type. Fundamentally it takes advantage of L. Schwartz's theorem that a distribution that is a generalized solution of an elliptic differential equation with sufficiently differentiable coefficients is a regular function. This yields theorem 3. If

$$L=\sum_{0\le k_1+\cdots+k_n\le r}a_{k_1\cdots k_n}(x)\frac{\partial^{k_1+\cdots+k_n}}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}}$$

is elliptic, the coefficients $k_1+\cdots+k_n+p$ times ($p\ge 2n+r$) differentiable, and A a self-adjoint operator corresponding to L, then there exists a kernel

$$\gamma(x,y,z)\quad(x,y\in E^n,\ x\ne y,\ z\in\sigma(A)),$$

that is p-times differentiable with respect to x, y, behaves like $|x-y|^{r-n}$ for $n>r$ ($\log|x-y|$ for $n=r$) and is bounded for $n<r$, such that $(R_z(A)f)(x)=\int\gamma(x,y;z)f(y)dy$ for any bounded finite function $f(y)$. For almost all λ (with respect to the measure ρ) there exists a $\psi(x,y,\lambda)$, p times differentiable, positive definite, such that for the spectral measure E of A

$$(E_\Delta f,g)=\int_\Delta d\rho(\lambda)\iint\psi(x,y,\lambda)f(y)\overline{g(x)}dxdy$$

for arbitrary finite functions $f,g\in L^2$ and arbitrary finite or infinite Δ. Further $L_x(\psi)=L_y(\psi)=\lambda\psi$.

František Wolf (Berkeley, Calif.)

3647:

Kac, G. I. Expansion in characteristic functions of self-adjoint operators. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 19–22. (Russian)

Let T be a closed operator in the Hilbert space H, with dense domain D_T and bounded inverse T^{-1}. T^{-1} is required to admit a bounded extension to all of H with finite H-norm. H_T denotes the adjoint space to D_T when the latter is normed by $\|\varphi\|^*=\|T\varphi\|$. Let A be an arbitrary self-adjoint operator with resolution of the identity $E(\Delta)$ and spectral function $\sigma(\Delta)$. The author shows that for each $f\in H$ there corresponds a subset Λ_f of the reals whose complement has σ measure zero and such that for $\lambda\in\Lambda_f$ and $\varphi\in D_T$, $\lim(\varphi, E(\Delta_\lambda^{(n)}f))/\sigma(\Delta_\lambda^{(n)})$ defines an element f_λ of H_T. For each n, $\Delta_\lambda^{(n)}$ is the interval $[k/n, k-1/n)$ containing the point λ. Under the same assumptions there exists for almost all λ a positive operator k_λ having finite H-norm such that $(\varphi,\varphi')=\int(k_\lambda T\varphi, T\varphi')d\sigma_\lambda$. The above results provide a generalization of results of U. M. Berezanski [article reviewed above] and I. M. Gelfand and A. G. Kostyučenko [Dokl. Akad. Nauk SSSR **103** (1955), 349–352; MR **17**, 388].

A. N. Milgram (Berkeley, Calif.)

3648:

Žautykov, O. A. Generalized Poisson bracket for functions of an even number of variables. Mat. Sb. N.S. **43** (85) (1957), 29–36. (Russian)

By functions of an even number of variables the author means a function $g(x_1,x_2,\cdots;z_1,z_2,\cdots)$ where the number of x's and z's, finite or infinite, is the same. After making suitable uniformity assumptions, the Poisson bracket is defined by

$$(f,g)=\sum\left(\frac{\partial f}{\partial x_k}\cdot\frac{\partial g}{\partial z_k}-\frac{\partial f}{\partial z_k}\cdot\frac{\partial g}{\partial x_k}\right)$$

and the series is shown to be uniformly convergent. The usual properties of the Poisson bracket, including the Poisson identity for three functions, are shown to hold. This permits the definition of a system in involution and the integrability conditions to be formulated for a finite system of equations of the form

$$X_k(z)=\sum a_{ks}(x_1,\cdots)\partial z/\partial x_s=0\quad(k=1,2,\cdots,m)$$

in the denumerable set of derivatives $\partial z/\partial x_s$ $(s=1,2,\cdots)$.

A. N. Milgram (Berkeley, Calif.)

3649:

Dezin, A. A. Concerning solvable extensions of the first order partial linear differential operators. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 11–14. (Russian)

3650:

Douglis, Avron. An ordering principle and generalized solutions of certain quasi-linear partial differential equations. Comm. Pure Appl. Math. **12** (1959), 87–112.

On considère l'équation $u_t+(F(u))_x=0$ avec la condition initiale $u(x,0)=\bar{u}(x)$. On appelle solution faible une fonction u satisfaisant à la relation $\oint(-udx+F(u)dt)=0$ pour tout chemin fermé régulier dans le semiplan $t\geq 0$. La solution faible est appelée solution généralisée si $u(x-0,t)\geq u(x+0,t)$ ou $u(x+h,t)-u(x,t)\leq h/at$ quand F est strictement convexe, et normalisée si $u(x,t)=u(x-0,t)$.

Si F est strictement convexe, la relation $u\leq v$ pour les conditions initiales implique la même relation pour les solutions généralisées, bornées et normalisées. De ce théorème on déduit un théorème d'unicité. L'A. donne aussi un théorème d'existence et une méthode de construction. *G. Marinescu* (Bucharest)

3651:

Rajagopal, A. K. A note of Ballabh's paper. Gaṇita **9** (1958), 5–7.

R. Ballabh [Gaṇita **5** (1954), 93–96; MR **18**, 902] examined a particular type of partial differential equation of the second order which can be reduced to Laplace's equation by means of the transformation $\rho=\rho(r)$, $r^2=x^2+y^2$, or $r^2=x^2+y^2+z^2$, respectively. The present note is an extension to a more general type of partial differential equation. *I. A. Barnett* (Cincinnati, Ohio)

3652:

Nilsson, Nils. Essential self-adjointness and the spectral resolution of Hamiltonian operators. Kungl. Fysiogr. Sällsk. i Lund Förh. **29** (1959), no. 1, 1–19.

Consider the Schrödinger operator

$$[H_0u](x) = -\nabla^2u(x)+V(x)u(x),$$

∇^2 being the Laplacian, with initial domain say of all complex-valued functions u over $x\in R_n$, euclidean n-space, which possess continuous partials everywhere of all orders and which vanish outside bounded sets of R_n. Assuming the fixed (potential) function $V(x)$ to be real valued and measurable over $x\in R_n$ and to be finitely square integrable over every finite interval of R_n, then of course H_0 is a symmetric operator in the Hilbert space $L_2(R_n)$. The author proves here that H_0 is essentially self-adjoint, equivalently (since H_0 is real), that it possesses a unique self-adjoint extension, if V satisfies the additional condition (somewhat simplified) that it be a finite sum of real measurable V_k, for each of which and each $y\in R_n$ there exists a non-singular linear transformation T_y on R_n, having the matrix elements of both T_y and T_y^{-1} bounded over $y\in R_n$, such that for some integer m, $0\leq m\leq n$, and real α satisfying $\alpha>m/2$ and $\alpha\geq 2$ the function

$$\int_A |V_k(y+T_y(z))|^\alpha dz_1dz_2\cdots dz_m$$

be bounded over $y\in R_n$ and $|z_j|\leq 1$ for $m+1\leq j\leq n$, where we define

$$A = \{(z_1,\cdots,z_m)\in R_m|\ |z_j|\leq 1 \text{ for } 1\leq j\leq m\}.$$

This greatly strengthens the result of Kato [Trans. Amer. Math. Soc. **70** (1951), 195–211; MR **12**, 781] proving the same conclusion for a special case where $m=3$, $\alpha=2$, $n=3p$ for some integer $p\geq 1$, $T_y=T$ has very special form, and the integral over A is independent of $z_{m+1},\cdots,z_n$. Essentially the special case of the author's theorem where $m=n$ was proved independently by the reviewer at about the same time [Pacific J. Math. **9** (1959), 953–973; theorem T.1 of sec. 2] by merely extending Kato's original argument dealing with Fourier transforms by use of the Young-Hausdorff-Titchmarsh theorem. The author's argument is closer to the work of Stummel [Math. Ann. **132** (1956), 150–176; MR **19**, 283], and passes from the case $m=n$ to $0\leq m\leq n$ by use of a lemma of Krylov's. *F. H. Brownell* (Seattle, Wash.)

3653:

Fox, David W.; and Pucci, Carlo. The Dirichlet problem for the wave equation. Ann. Mat. Pura Appl. (4) **46** (1958), 155–182. (English summary)

The authors investigate the Dirichlet problem for $u_{xx}-u_{tt}=0$ in the rectangle R $[0\leq x\leq\pi,\ 0\leq t\leq T]$ with the boundary data $u(0,t)=u(\pi,t)=0$, $u(x,0)=\varphi(x)$ and $u(x,T)=\psi(x)$. They establish a correspondence between this problem and the following functional equation (*) for an even, continuous function E of period 2π: $E(x+T)-E(x-T)=F(x)$, where F is a prescribed continuous odd periodic function of period 2π. Existence and uniqueness of the solution are discussed by considering the corresponding questions concerning (*). Formal expressions for the solution are deduced. After having shown by examples that the solution does not depend continuously on the data the authors investigate what information about the solution can be obtained when the data are known approximately. In particular, they examine this question under the assumption that the derivatives of the solution are—a priori—bounded. *A. Huber* (Muenchenstein)

3654:

Arutyunyan, N. H.; Džrbašyan, M. M.; and Aleksandryan, R. A. On a method of solution for a hyperbolic equation with a mixed derivative. Izv. Akad. Nauk Armyan. SSR Ser. Fiz.-Mat. Nauk **10** (1957), no. 1, 113–121. (Russian. Armenian summary)

The equation

$$\partial^2u/\partial t^2 = \partial^2u/\partial x^2-2a\partial^2u/\partial x\partial t$$

with boundary conditions

$$u(x,0)=f(x),\quad u_t(x,0)=g(x),\quad u(0,t)=u(1,t)=0$$

is shown to have a unique solution which is regular when f and g are independent functions piece-wise continuously differentiable up to the fourth and third orders, respectively, and

$$f(0)=f''(0)=f'''(0)=f(1)=f''(1)=f'''(1);$$
$$g(0)=g'(0)=g''(0)=g(1)=g'(1)=g''(1).$$

A. N. Milgram (Berkeley, Calif.)

3655:

Ciliberto, Carlo. Precisazione relativa alla memoria: Sulle equazioni non lineari di tipo parabolico in due variabili. Ricerche Mat. 7 (1958), 232–234.

L'A. completa la dimostrazione di un teorema esposto in un suo precedente lavoro [Ricerche Mat. **3** (1954), 129–165; MR **16**, 1028]. Egli sostituisce una maggiorazione (formula (71)) con una più precisa e indica le modifiche da apporre successivamente. *G. Prodi* (Trieste)

3656a:

Pogorzelski, W. Propriétés des intégrales de l'équation parabolique normale. Ann. Polon. Math. **4** (1957), 61–92.

3656b:

Pogorzelski, W. Problèmes aux limites pour l'équation parabolique normale. Ann. Polon. Math. **4** (1957), 110–126.

It is proved that a solution $u(x, t)$ exists for a quite general class of mixed-initial-boundary-value problems for the normal parabolic equation

$$\hat{\psi}(u) \equiv \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + \sum_{\alpha=1}^{n} b_\alpha \frac{\partial u}{\partial x_\alpha} + cu - \frac{\partial u}{\partial t} = F(x, t, u)$$

when the coefficients $a_{\alpha\beta}$, b_α, c and F are Hölder-continuous. The result depends on a previous construction by the same author [Ricerche Mat. **5** (1956), 25–57; MR **18**, 47] of a fundamental solution of $\hat{\psi}(u)=0$. The first of the present paper defines "generalized potentials" for the equation $\hat{\psi}=0$ which are analogous to the potentials connected with elliptic equations; some boundedness and Hölder-continuity properties of these potentials are proved. The second paper employs these potentials, together with the Schauder fixed point theorem, to construct a solution of the initial-boundary-value problem.

H. C. Kranzer (Garden City, N.Y.)

3657:

Friedman, Avner. On the uniqueness of the Cauchy problem for parabolic equations. Amer. J. Math. **81** (1959), 503–511.

Consider the equation $\partial u(x, t)/\partial t = Lu(x, t)$ where L is uniformly elliptic on E_n (Euclidean space) for $0<t<d$, L is of order 2, and the coefficients of L satisfy the regularity conditions required for F. G. Dressel's fundamental solution [Duke Math. J. 7 (1940), 186–203; MR **2**, 204]. The author proves that a solution of the equation which vanishes for $t=0$ is identically 0 provided that

$$\int_0^d \int_{E_n} \exp\{-K|x|^2\}|u(x, t)|dxdt < \infty,$$

where $K>0$ (theorem 1) or that $u(x, t) \geqq 0$ (theorem 2). The proof of theorem 1 generalizes to higher order systems provided they are parabolic in the sense of Petrovskiĭ and such that S. D. Èĭdel'man's fundamental solution [Mat. Sb. (N.S.) **38 (80)** (1956), 51–92; MR **17**, 857] exists, upon replacing $|x|^2$ by $|x|^{2m/(2m-1)}$, where $2m$ is the order of the system. *E. Nelson* (Princeton, N.J.)

3658:

Devingtal', Yu. V. Existence and uniqueness of the solution of the Frankl' problem. Uspehi Mat. Nauk **14** (1959), no. 1 (85), 177–182. (Russian)

For the equation of mixed type

$$\operatorname{sign} y \cdot |y|^m \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0 \quad (m > 0)$$

or an equivalent first order system the author considers the following special case of the Frankl' problem. Let the domain D be bounded by (i) the straight line segment from $A(0, 1)$ to $A'(0, -1)$; (ii) the arc of $x^2+(1-2\beta)^2 y^{m+2} = (1-2\beta)^2$, $\beta = m/(2m+4)$, from A to its intersection with $y=0$, $x>0$; and (iii) the characteristic arc $A'C$. Impose boundary conditions (i) $u=0$ on AC; (ii) $\partial u/\partial x = 0$ on AA'; (iii) $u(0, y)-u(0, -y)=f(y)$ on AA'. In the closure of D, $\partial u/\partial x$ and $\partial u/\partial y$ are to be continuous except, possibly, for infinities of order less than one at (0, 0), A, A', and C. First the author reflects D with respect to the y-axis and continues u into $x<0$ by $u(-x, y)=u(x, y)$. In terms of $u(x, 0)=\tau(x)$ and $\partial u(x, 0)/\partial y = \nu(x)$ he represents u in $y<0$ by quadratures as the solution of the singular Cauchy problem. In the symmetrical region $y>0$ he finds u in terms of $\nu(x)$ by an integral representation due to E. Holmgren [Ark. Mat. Astr. Fys. **19B** (1926), 1–16]. These representations and boundary conditions (iii) yield a singular integral equation for $\nu(x)$ which can be transformed by a device due to S. G. Mihlin [Uspehi Mat. Nauk (N.S.) **3** (1948), no. 3 (25), 29–112; MR **10**, 305] into a Fredholm integral equation with bounded, continuous kernel. Uniqueness of the solution of the Frankl' problem has previously been established by Frankl' [Akad. Nauk SSSR. Prikl. Mat. Meh. **20** (1956), 192–202; MR **18**, 255].

J. H. Giese (Aberdeen, Md.)

3659:

Gillis, Paul P. Propriétés et existence des solutions de certaines classes d'équations du type elliptique. Bull. Soc. Math. France **86** (1958), 283–297.

A review of the recent literature concerning Dirichlet's problem for linear and nonlinear elliptic partial differential equations in two or more independent variables.

P. R. Garabedian (New York, N.Y.)

3660:

Schechter, Martin. General boundary value problems for elliptic partial differential equations. Bull. Amer. Math. Soc. **65** (1959), 70–72.

If A is a linear elliptic differential operator of order $2r$ in a bounded domain $G \subset R^n$ and $\{B_j\}$ is a normal system of r boundary operators at the boundary of G, then the boundary value problem $\pi(A, f, u_0, \{B_j\})$ is that of finding a function $u \in C^\infty(\bar{G})$ such that $Au=f$ in G and $B_j u = B_j u_0$, $j=1, 2, \cdots, r$, on the boundary of G.

The theorem of this note states (without proof) that under certain algebraic conditions on the characteristic polynomials of A and the B_j the boundary value problem $\pi(A, f, u_0, \{B_j\})$ has a solution for every f and u_0 if and only if the solution of $\pi(A^*, 0, 0, \{B_j^*\})$ is unique. Here A^*, $\{B_j^*\}$ is the differential system adjoint to A, $\{B_j\}$ in the sense of Aronszajn and Milgram [Rend. Circ. Mat. Palermo (2) **2** (1953), 266–325; MR **16**, 252].

K. T. Smith (Madison, Wis.)

3661:

Jordán Díaz, Plácido. Vectorial method of obtaining the characteristic curves of a system of quasi-linear partial differential equations of hyperbolic type. Rev. Soc. Cubana Ci. Fís. Mat. 4 (1957/58), 91–94. (Spanish)

En partant de deux équations aux dérivées partielles

linéaires des fonctions u et v et variables x, y dont les coefficients sont fonctions de u, v, x, y, l'auteur arrive par quelques transformations élémentaires à obtenir l'équation des courbes caractéristiques. *M. Kiveliovitch* (Paris)

3662:

Duff, G. F. D. Mixed problems for hyperbolic equations of general order. Canad. J. Math. **11** (1959), 195–221.

The object of the paper is the extension to linear partial differential equations of order m in N independent variables, of the existence theorems for mixed initial and boundary value problems which the author had previously established for systems of first-order equations [same J. **10** (1958), 127–160; MR **20** #4071]. In such mixed problems an initial surface S and a boundary surface T are the carriers of two types of data, and the number of datum functions to be assigned on T depends upon the configuration of the characteristic surfaces relative to S and T. In the case of analytic coefficients and data, and under a condition weaker than the usual regular hyperbolic condition, the single equation of higher order is reduced to a first-order system of the type treated in the above-cited paper. For this purpose a certain algebraic lemma, related to the characteristic roots, is required. The non-analytic problem is treated by the energy integral method. A sufficient condition is given for existence. Since this condition seems not always fulfilled, two special cases are treated in detail. In the first case there is one less boundary condition than initial conditions; in the second there are half as many, and in addition an assumption of symmetry relative to the boundary surface. (Adapted from the author's introduction) *R. W. McKelvey* (Boulder, Colo.)

3663:

Cinquini-Cibrario, Maria. Sistemi di equazioni a derivate parziali in più variabili indipendenti. Ann. Mat. Pura Appl. (4) **44** (1957), 357–417.

The author considers the Cauchy problem for a quasi-linear system of hyperbolic differential equations

$$(*)\quad \frac{\partial z_i}{\partial x}+\sum_{r=1}^{h}\rho_{ir}(x, y_1, \cdots, y_h; z_1, \cdots, z_m)\frac{\partial z_i}{\partial y_r}= f_i(x, y_1, \cdots, y_h; z_1, \cdots, z_m),$$
$$z_i(0, y_1, \cdots, y_h)=\Phi_i(y_1, \cdots, y_h)$$

($i=1, 2, \cdots, m$) under the following assumptions: The functions ρ_{ir} and f_i are defined in a domain

$$D_{a_0}: \{0\leq x\leq a_0,\ -\infty<y_1<\infty, \cdots, -\infty<z_m<\infty\}.$$

For fixed x, they are continuous in $y_1, \cdots, z_m$, and for fixed $y_1, \cdots, z_m$, they are quasi-continuous in x. The absolute values of ρ_{ir} and f_i and their Lipschitz quotients are majorized by non-negative quasi-continuous integrable functions of x. The functions $\Phi_i(y_1, \cdots, y_h)$ are defined and Lipschitzian for all values of $y_1, \cdots, y_h$. The—generalized—solutions are required to be absolutely continuous in x and Lipschitz continuous in $y_1, \cdots, y_h$. They must satisfy the differential equations almost everywhere and the initial conditions everywhere. The author proves that there is a number $a\leq a_0$ such that the Cauchy problem (*) has uniquely determined generalized solutions in D_a. They depend continuously on the initial data. Furthermore conditions are given under which the solutions will have continuous partial y_r-derivatives everywhere in D_a. Then, if the functions ρ_{ir} and f_i are continuous, the generalized solutions will be solutions in the ordinary sense. For the proofs an equivalent system of integral equations is introduced. For its solution use is made of results of the theory of ordinary differential equations and an interesting inequality. *J. C. C. Nitsche* (Minneapolis, Minn.)

3664:

Nagumo, Mitio. On linear hyperbolic system of partial differential equations in the whole space. Proc. Japan Acad. **32** (1956), 703–706.

This article contains an existence theorem for the Cauchy problem for operator equations of the form $u_t-Au=0$, where the operator A is assumed to be half bounded with respect to the norm induced by some symmetric, positive definite and invertible operator Λ; i.e., we assume $(Au, \Lambda u)\leq c(u, \Lambda u)$. The construction uses the representation of linear functionals continuous with respect to the dual of the norm induced by Λ. This abstract existence theorem is modeled after, and is applicable to, the theory of hyperbolic equations. *P. D. Lax* (New York, N.Y.)

3665:

Shirota, Taira. The initial value problem for linear partial differential equations with variable coefficients. I. Proc. Japan Acad. **33** (1957), 31–36.

The author considers the initial value problem for the equation $u_t-Au=0$, where A is a matrix-differential operator with variable coefficients semi-bounded with respect to another differential operator B which is positive definite. More precisely, this is assumed to hold for a sequence of operators B^s of increasing degree. It is shown that for each t the resolvent of A exists and satisfies the hypotheses of the Hille-Yosida theorem. From this it is shown, by suitably modifying Yosida's method, that the Cauchy problem is well posed. Applications to parabolic, hyperbolic and the Schrödinger equation are given. *P. D. Lax* (New York, N.Y.)

3666:

Shirota, Taira. The initial value problem for linear partial differential equations with variable coefficients. II. Proc. Japan Acad. **33** (1957), 103–104.

This note contains another proof for the existence theorem of the previous note, based on a duality argument in Hilbert space, somewhat as in Nagumo's article [#3664 above]. *P. D. Lax* (New York, N.Y.)

3667:

Shirota, Taira. The initial value problem for linear partial differential equations with variable coefficients. III. Proc. Japan Acad. **33** (1957), 457–461.

Let Ω be a domain in Euclidean space, V a subspace of $L_2(\Omega)$ containing all smooth compactly carried functions, and $(u, v)_t$ a time dependent bilinear form over v positive definite and continuous, and Lipschitz continuous in t. There is then induced a time dependent operator A, whose domain includes boundary conditions built into the space V. In this paper the initial value problem for $u_t-Au=0$ is solved. It is shown that the Fokker-Planck equation is included in the class of equations discussed above. *P. D. Lax* (New York, N.Y.)

3668:

Shirota, Taira. On Cauchy problem for linear partial differential equations with variable coefficients. Osaka Math. J. **9** (1957), 43–59.

In this paper the Cauchy problem is solved for the equation $u_t - Au = v$ where A is a differential operator semi-bounded in the scalar product induced by another—symmetric and positive definite—differential operator B. Solutions are constructed by orthogonal projection in suitable Hilbert spaces. Applications are given to very general classes of hyperbolic and parabolic equations.

P. D. Lax (New York, N.Y.)

3669:

Phillips, R. S. Dissipative operators and hyperbolic systems of partial differential equations. Trans. Amer. Math. Soc. **90** (1959), 193–254.

An operator L on a Hilbert space H_0 is said to be dissipative if $\langle Ly, y\rangle + \langle y, Ly\rangle \leqq 0$ for all y in the domain of L. It is maximal dissipative if it is not the proper restriction of another dissipative operator. Theorem: an operator is the infinitesimal generator of a strongly continuous semi-group of contraction operators on H_0 if and only if it is maximal dissipative and has a dense domain.

With this theorem the author begins a detailed study of dissipative operators and their maximal dissipative extensions. As in a previous paper [same Trans. **86** (1957), 109–173; MR **19**, 863] his principal aim is to characterize the well-set Cauchy problem for a hyperbolic system of the form $y_t = L_1 y$, where y is a vector-valued function of time and several space variables and L_1 is a matrix of time-independent first order differential operators. Here one has at hand the natural concept of an energy integral which will serve as an inner product, and the search for problems in which the energy is never increasing now becomes a search for maximal dissipative operators. It is assumed that L_1 is at least locally dissipative so that the search is reduced to finding proper boundary data. The operator L_1 is defined more precisely by saying in effect that it has the largest domain for which a boundary integral can be made meaningful. A second operator L_0 is defined to be the largest restriction of L_1 whose domain consists of functions which are essentially zero on the boundary. Then all maximal dissipative operators L such that $L_0 \subset L \subset L_1$ are completely characterized. To include boundary conditions of the elastic type the system is 'coupled' to a second system. This second system is represented when uncoupled by a motion whose infinitesimal generator is a bounded operator on a second Hilbert space $\dot{H}_0$. The coupling takes the form of a linear transformation from the space of boundary data to $\dot{H}_0$. After putting restrictions of a dissipative nature on this coupled system, the maximal dissipative operators are again characterized in the same sense as before.

The paper is divided into two chapters. Chapter I considers the purely abstract formulation, while Chapter II applies the results of Chapter I to the systems mentioned above. Finally an appendix considers extensions of L_0 which are maximal dissipative but which are not restrictions of L_1. A particular class of such extensions is constructed.

G. Hufford (Seattle, Wash.)

3670:

Dezin, A. A. Mixed problems for certain parabolic systems. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 503–506. (Russian)

3671:

Manžeron, D. [Mangeron, D.] Sopra i problemi di Dirichlet per una classe di equazioni alle derivate "totali". Bul. Inst. Politehn. Iaşi (N.S.) **3** (1957), 49–52. (Russian. Italian and Romanian summaries)

Folgendes Eigenwertproblem wird betrachtet:

$$\partial^{2n}u/\partial x^n\partial y^n + \lambda A(x, y)u = 0 \quad (n = 2, 3, \cdots),$$

wobei A auf $a \leqq x \leqq c$, $b \leqq y \leqq d$ stetig sein soll, mit den Randbedingungen $\partial^{2k}u/\partial x^k\partial y^k = 0$ $(k = 0, 1, 2, \cdots, n-2)$ auf $x = a$ und $y = b$ sowie $u = 0$ auf $x = c$ und $y = d$. Der Verf. gibt eine dazu äquivalente Fredholmsche Integralgleichung an (ohne Beweis).

A. Huber (Münchenstein)

3672:

Lewy, Hans. Composition of solutions of linear partial differential equations in two independent variables. J. Math. Mech. **8** (1959), 185–192.

Let $L[v] = \sum_{k+l\leq p} a_{kl}(\partial^{k+l}v/\partial x^k\partial y^l)$ be a linear partial differential operator with constant coefficients and let $M[v]$ be its adjoint. By Green's formula, we define two linear forms $B_i(u, v)$ $(i = 1, 2)$:

$$\iint_D (uL[v] - vM[u])dxdy = \int_D (B_1(u, v)dx + B_2(u, v)dy).$$

The B_i are not uniquely determined by this definition; we select an arbitrary but fixed pair of such forms. The author proves the theorem: Let $L[u] = 0$ and $L[v] = 0$; then the line integral

$$w(x, y) = \int_{0,0}^{x,y} (B_1[u(x-\xi, y-\eta), v(\xi, \eta)]d\xi \\ + B_2[u(x-\xi, y-\eta), v(\xi, \eta)]d\eta)$$

is independent of the path of integration and satisfies $L[w] = 0$. Thus, from any two solutions of the differential equation a new solution can be obtained by this process of combination. The proof is obtained by a lengthy but straightforward verification.

M. Schiffer (Stanford, Calif.)

3673:

Hörmander, Lars. Differentiability properties of solutions of systems of differential equations. Ark. Mat. **3** (1958), 527–535.

La première partie est consacrée à une caractérisation des systèmes hypo-elliptiques. Soit $\mathfrak{P}(D)u = f$ un système d'équations aux dérivées partielles, où $u = (u_1, \cdots, u_n)$ et $f = (f_1, \cdots, f_m)$; $\mathfrak{P}(D)$ est une matrice à m lignes et n colonnes dont les éléments sont des opérateurs différentiels à coefficients constants définis dans R^ν. Soit $\mathfrak{P}(\zeta)$ la matrice déduite de $\mathfrak{P}(D)$ par la substitution $\partial/\partial x_j \to i\zeta_j$ $(j = 1, \cdots, \nu)$. Alors, pour que $\mathfrak{P}(D)$ soit hypo-elliptique il faut et il suffit que $m \geqq n$ et que la distance du point réel ξ à l'ensemble des $\zeta \in C^\nu$ où le rang de $P(\zeta) < n$ tende vers infini avec ξ.

La nécessité est presque évidente, vu la démonstration donnée pour un seul opérateur par l'auteur dans sa thèse. La suffisance est une conséquence immédiate du travail de Lech cité plus haut [#3542].

La seconde partie est consacrée aux évaluations des dérivées successives des solutions des systèmes hypoelliptiques, et monte que ces évaluations ne peuvent pas être améliorées. *S. Mizohata* (Kyoto)

3674:

Hörmander, Lars. On the uniqueness of the Cauchy problem. Math. Scand. **6** (1958), 213–225.

In some problems of Cauchy, uniqueness has been proved by the aid of inequalities of the form

$$(*)\quad \tau^\gamma \int |Qu|^2 e^{2\tau\varphi}dx \le C \int |Pu|^2 e^{2\tau\varphi}dx,\quad u\in C_0^\infty(\Omega),\ \tau\ge 1,$$

where P and Q are two linear differential operators (or systems), Ω is an open set, φ a fixed function, and C and γ are constants independent of u and τ. Here, by use of an inequality of Treves [Thèse, Paris, 1958, mimeographed], general precise conditions on P and Q are obtained for the existence of a (non-linear) function φ such that (*) will hold. A homogeneous P, for example, and any Q of lower order are subject to (*) if, and only if, the real characteristics of P are simple and the complex characteristics at most double. A systematic attempt is not made to apply the general results in uniqueness questions, but the scope of these results is illustrated by the following deduction: Let u be a classical solution of an equation $P(x, D)u=0$ with continuous coefficients, which is defined in a neighborhood of 0, and let $u=0$ outside of a sphere passing through 0. Suppose that the principal part of $P(x, D)$ has constant coefficients and that its real characteristics are simple and its complex characteristics at most double. Then u vanishes in a neighborhood of 0.
A. Douglis (College Park, Md.)

3675:

Calderón, A. P. Uniqueness in the Cauchy problem for partial differential equations. Amer. J. Math. **80** (1958), 16–36.

The problem of distinguishing all the types of partial differential equations, whose solutions are uniquely determined by Cauchy data, is of widespread current interest. In the paper under review, considerable advances are made in the general solution of this problem on the hypothesis that the (real and complex) characteristics of the equations considered are non-multiple. Linear equations first are considered, then systems of linear equations, and finally non-linear equations and systems reduced by well known methods to the linear case. The smoothness required of the coefficients in the linear cases, and of the differential equations in the non-linear cases, is light. Uniqueness in Cauchy's problem then is established for second and third order equations in any number of independent variables [the second order case is already known from Aronszajn, J. Math. Pures Appl. (9) **36** (1957), 235–249; MR **19**, 1056], for equations of any order not in three independent variables, and for systems of equations not in three or four independent variables. The method is based on the fact that a linear differential operator Au of homogeneous order m defined on all Euclidean space and with bounded coefficients is representable as $Au=H\Lambda^m u$, where Λ is a square root of the negative of the Laplacian, and H is a singular integral operator. This fact, in the light of properties of Λ and H previously established by Calderón and Zygmund [Amer. J. Math. **79** (1957), 901–921; MR **20** #7196], makes possible the reduction of the uniqueness problem to a relatively simple form.
A. Douglis (College Park, Md.)

3676:

★**Fantappiè, Luigi. Sur les méthodes nouvelles d'intégration des équations aux dérivées partielles au moyen des fonctionnelles analytiques.** La théorie des équations aux dérivées partielles. Nancy, 9–15 avril 1956, pp. 47–62. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXI. Centre National de la Recherche Scientifique, Paris, 1956. 187 pp. 1500 francs.

The author gives a new method for the solution of Cauchy's problem. The work constitutes an improvement of the fourth of five methods previously given by him [Second colloque sur les équations aux derivées partielles, Bruxelles, 1954, pp. 95–128, Thone, Liège; Masson, Paris; 1955; MR **18**, 806]. The present method uses the parametric representation of the characteristic cone of the equation. The work is carried through for an equation in three variables in which all terms are of the same order m:

$$Lu=\frac{\partial^m u}{\partial t^m}+\sum a_{r_0r_1r_2}\frac{\partial^m u}{\partial t^{r_0}\partial x^{r_1}\partial y^{r_2}}=f(t,x,y),$$

where the coefficients $a_{r_0r_1r_2}$ are constants, $r_0+r_1+r_2=m$, $r_0<m$, with initial conditions $u_t^{(j)}(0,x,y)=\phi_j(x,y)$, $0\le j\le m-1$. All given functions are analytic. A further hypothesis is that the characteristic cone of the equation has at most double tangent planes and ordinary inflection planes. There is a section on equations invariant under a group with particular mention of various relativistic groups.
E. R. Lorch (New York, N.Y.)

3677:

Budak, B. M. On the straight line method for certain boundary problems. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 9–12. (Russian)

3678:

Budak, B. M. On the method of straight lines for certain boundary problems involving systems of partial differential equations. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 187–190. (Russian)

3679:

Slezinger, I. N. On a method of solution of linear boundary problems of self-conjugate type. Prikl. Mat. Meh. **20** (1956), 704–713. (Russian)

The usual formulation is introduced for the variational energy method for a class of self-adjoint differential equations of arbitrary even order. In connection with results known in the literature [S. Zaremba, Cracow Acad. Sci. Bull. Internat. Cl. Sci. Mat. Nat. **1908**, no. 1, 1–29; M. G. Kreĭn, Mat. Sb. N.S. **20** (**62**) (1948), 431–495; MR **9**, 515; and elsewhere] it is noted that knowledge of the solution of one boundary problem can in some cases facilitate solution of other boundary problems for the same differential equation. These ideas are applied to computation for the clamped rectangular plate under certain types of loading with utilization of the solution for the hinge-supported plate.

The author notes the connection of his approach with the variational method proposed for the biharmonic problem by È. H. Rafal'son [Dokl. Akad. Nauk SSSR **64** (1949), 799–802; MR **10**, 707].
M. Š. Bitman (RŽMat **1957** #7887)

3680:
Lions, Jacques-Louis. Sur l'existence de solutions des équations de Navier-Stokes. C. R. Acad. Sci. Paris **248** (1959), 2847–2849.

Let D_t^γ be the fractional derivative with respect to t of order γ, $0<\gamma<\frac{1}{4}$. Using Fourier transforms, the author establishes the a priori estimate

$$\int_0^\infty |D_t^\gamma u(t)|^2 dt \leqq \text{const}\cdot(J+J^{3/2}),$$

where u is a solution of the Navier-Stokes equation in a region of R^n, $n\leqq 4$, $|\ |$ denotes the L^2 norm, and $J=\int_0^\infty |f(t)|^2dt+|u(0)|^2$, f denoting the external forces. This estimate is strong enough to ensure existence of weak, or "turbulent", solutions for all time with $\int_0^\infty |D_t^\gamma u(t)|^2dt<\infty$. This is greater regularity than that established by J. Leray [Acta Math. **63** (1934), 193–248] for turbulent solutions. The uniqueness question remains open.
E. Nelson (Princeton, N.J.)

POTENTIAL THEORY

See also 3538, 3549, 3656a–b, 3701, 3887.

3681:
Arsove, Maynard G. A new proof of the convergence theorem for δ-subharmonic functions. Illinois J. Math. **3** (1959), 217–221.

Le théorème de convergence en question avait été établi antérieurement par l'auteur [Trans. Amer. Math. Soc. **75** (1953), 526–551; MR **15**, 622; il s'agit du théorème 34] pour deux dimensions; une simplification de la méthode permet des généralisations diverses.
J. Deny (Palaiseau)

3682:
Nozaki, Yasuo. On generalization of Frostman's lemma and its applications. Kōdai Math. Sem. Rep. **10** (1958), 113–126.

Ce que l'auteur appelle généralisation d'un lemme de Frostman n'est autre que la formule de composition de M. Riesz dans R^m:

$$r^{-k} * r^{-l} = H_m(k, l) r^{m-k-l} \qquad (0<k<m,\ 0<l<m,\ m<k+l).$$

Remarques sur la convergence et la continuité de diverses intégrales liées à la théorie des potentiels de Riesz [Acta Sci. Math. Szeged **9** (1938), 1–42]. *J. Deny* (Palaiseau)

3683:
Heinz, Erhard. On one-to-one harmonic mappings. Pacific J. Math. **9** (1959), 101–105.

The paper deals with one-to-one harmonic mappings $z=z(w)$ $(z=x+iy,\ w=u+iv)$ of the disc $|w|<1$ onto the disc $|z|<1$ such that $z(0)=0$. Although, due to a lemma of H. Lewy [Bull. Amer. Math. Soc. **42** (1936), 689–692], the Jacobian $\Delta=\partial(x,y)/\partial(u,v)=|z_w|^2-|z_{\bar{w}}|^2$ cannot vanish, unfortunately the greatest lower bound of Δ for all such mappings is zero. However, as author has shown before [Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1952**, 51–56; MR **14**, 885], the weaker statement $|z_u(0)|^2+|z_v(0)|^2\geqq\mu>0$ is true. The best value of his universal constant μ is not known yet. (The best value known so far, given by the reviewer, is 0.64.) The author proves in the present paper the inequality $|z_u(w)|^2+|z_v(w)|^2\geqq 2/\pi^2$, valid for all $|w|<1$.
J. C. C. Nitsche (Minneapolis, Minn.)

3684:
Schiffer, M. The Fredholm eigen values of plane domains. Pacific J. Math. **7** (1957), 1187–1225.

The eigenfunctions of the classical Fredholm equation for solving Dirichlet's problem in the plane are expressed in a form exhibiting their connection with analytic functions in the interior and exterior domains associated with a closed curve. A dielectric Green's function is introduced and variational formulas are derived for this Green's function and for the Fredholm eigenvalues. As an application, a sharp estimate is found for the lowest non-trivial eigenvalue of a class of uniformly analytic curves.
P. R. Garabedian (New York, N.Y.)

3685:
Čerpakov, P. V. On convergence of series of functions that are harmonic in the limit. Kuĭbyšev. Aviacion. Inst. Trudy **2** (1954), 8–11. (Russian)

Let $u(M,t)$ be a solution of the heat equation $u_t=u_{xx}+u_{yy}+u_{zz}$ for $t\geqq 0$ and M on a region D. The notion of $u(M,t)$ harmonic in the limit is stated by the author only in a special case, but the general definition seems to be that $\lim_{t\to\infty} u(M,t)$ exists uniformly for M on compact subsets of D. (The limit function $u(M)$ is then harmonic on D.) Two theorems analogous to theorems on harmonic functions are established. Theorem 1: if a sequence of functions $u_n(M,t)$ harmonic in the limit converges uniformly in M on compact subsets of D for each $t\geqq 0$, then the limit function is harmonic in the limit. Theorem 2 is a similarly worded counterpart of the Harnack convergence theorem. *M. G. Arsove* (Seattle, Wash.)

3686:
Edwards, R. E. Cartan's balayage theory for hyperbolic Riemann surfaces. Ann. Inst. Fourier. Grenoble **8** (1958), 263–272, xii. (French summary)

This paper is concerned with the possibility of extending Cartan's result on the completeness of the space of positive measures of finite energy to the case of Riemann surfaces. This problem has been considered by several writers and the present author gives an explicit proof for the first time. Let X be a hyperbolic Riemann surface with Green's function g. The potential U^μ of a positive Radon measure μ on X is defined by $U^\mu(x)=\int g(x,y)d\mu(y)$, and the mutual energy (μ,ν) is defined by $\int U^\mu d\nu$. Let $\mathscr{E}$ denote the set of positive measures μ of finite energy (μ,μ). Using a primitive form of balayage theory it is first proved that $(\mu,\nu)^2\leqq(\mu,\mu)(\nu,\nu)$ for any positive measures μ and ν, and then the completeness of $\mathscr{E}$ with distance defined by

$$\|\mu-\nu\| = \sqrt{[(\mu,\mu)+(\nu,\nu)-2(\mu,\nu)]}$$

is proved. *M. Ohtsuka* (Hiroshima)

3687:

Reinhart, Bruce L. Harmonic integrals on almost product manifolds. Trans. Amer. Math. Soc. **88** (1958), 243–276.

Let V be a compact almost product manifold with torsionless metric. Let Δ be the Laplacian for differential forms, Δ' the Laplacian with respect to one of the two classes of almost product variables, and Δ'' correspondingly for the other. Set $\tilde{\Delta}=\Delta'+\Delta''$. The relationship between Δ and $\tilde{\Delta}$ is studied. A Green's operator is constructed for $\tilde{\Delta}$, but in general it does not commute with d, d', etc. For the operator Δ', a Green's operator G' is constructed, but with a restricted domain. For this restricted domain an isomorphism is established between the d' cohomology and the kernel of Δ'. A detailed example on the torus shows that for both the construction of G' and the isomorphism some such restriction is necessary.

M. P. Gaffney (Washington, D.C.)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 3546, 4054.

3688:

Wilf, Herbert S. Curve-fitting matrices. Amer. Math. Monthly **65** (1958), 272–274.

It is well known that the coefficients of the polynomial of degree $n-1$,

$$p(x) = a_0+a_1x+\cdots+a_{n-1}x^{n-1},$$

passing through the n equally spaced points (x_i, y_i), can be determined by the set of linear equations $V_na=y$, where

$$a = \{a_0, a_1, \cdots, a_{n-1}\}, \quad y = \{y_1, y_2, \cdots, y_n\}$$

and V_n is the Vandermonde matrix of order n.

Changing the origin and scale, by using the variable $=(x-x_1)/(x_n-x_1)$, the matrices V_n can be made independent of the values of x. The author gives the explicit form of the inverses for $n=2(1)6$.

D. C. Gilles (Glasgow)

3689:

Bakhvalov, N. S. On the number of arithmetical operations in solving Poisson's equation for a square by means of finite differences. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 252–254. (Russian)

3690:

Lass, Harry; and von Roos, Oldwig. The Dirac measure as applied to the solution of difference equations by means of transforms. Amer. Math. Monthly **66** (1959), 483–485.

The authors employ a generalization of the Dirac measure of L. Schwartz to obtain the solution of three particular homogeneous difference equations. Given the single-valued functions $f(x)$ and $\rho(x)$, define the measure $d\mu(x, y)$ in the xy-plane to be $d\mu=\rho(x)$ for $y=f(x)$, and $d\mu=0$ elsewhere. Define the integral of $g(x, y)$ with respect to $d\mu$ to be

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)d\mu = \int_{-\infty}^{\infty} g(x, f(x))\rho(x)dx$$

provided that the right-hand integral exists in the Riemann sense. Then for any suitable functions $\rho(x)$ and $F(x, y)$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y-f(x))F(x, y)d\mu = 0.$$

For example, to solve the equation $H_{\nu-1}(z)+H_{\nu+1}(z)+2H_\nu'(z)=0$, substitute

$$H_\nu(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\nu x}e^{-zy}d\mu$$

with $d\mu$ unspecified. The resulting expression

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\nu x}e^{-zy}\,(\cosh x-y)d\mu = 0$$

suggests defining $d\mu=\rho(x)$, $\rho(x)$ arbitrary, for $y=\cosh x$, and $d\mu=0$ elsewhere. This procedure provides the solution

$$H_\nu(z) = \int_{-\infty}^{\infty} \rho(x)e^{-\nu x}e^{-z\cosh x}\,dx$$

for each $\rho(x)$ for which the integral exists.

P. E. Guenther (Cleveland, Ohio)

3691:

Tverberg, Helge. A new derivation of the information function. Math. Scand. **6** (1958), 297–298.

Shannon's expression

$$H(x_1, x_2, \cdots, x_n) = c\sum x_i \log x_i$$

is derived from the following axioms on the function H: (i) H is defined for any set of non-negative arguments with sum 1, and it is symmetric in all arguments; (ii)

$$H(x_1, x_2, \cdots, x_{n-1}, u, v) = H(x_1, x_2, \cdots, x_n)+x_nH\left(\frac{u}{x_n}, \frac{v}{x_n}\right),$$

whenever all terms of the equation have a meaning; (iii) $H(x, 1-x)$ is integrable, in the sense of Lebesgue, on the interval $0\leq x\leq 1$.

Axiom (iii) is an improvement on the work of Fadiev and Khintchine, who assumed (i) and (ii) and the continuity of $H(x, 1-x)$.

S. W. Golomb (Pasadena, Calif.)

SEQUENCES, SERIES, SUMMABILITY

See also 3393, 3579, 3580, 3774.

3692:

Ney, A. Extension du domaine d'applicabilité du critère de convergence de d'Alembert. Gaz. Mat. Fiz. Ser. A **10 (63)** (1958), 709–712. (Romanian. French and Russian summaries)

The series $\sum u_n$ of positive terms converges if $u_{kq+r+1}/u_{kq+r}\to\lambda_r$ as $q\to\infty$ $(1\leq r\leq k)$, with $\lambda_1\lambda_2\cdots\lambda_k<1$, and diverges if the inequality is reversed.

R. P. Boas, Jr. (Evanston, Ill.)

3693:

Dawson, David F. Continued fractions with absolutely convergent even or odd part. Canad. J. Math. **11** (1959), 131–140.

For a continued fraction

$$f(a) = 1/1+a_1/1+a_2/1+\cdots$$

with $a_n \neq 0$ and with approximants $\{f_n\}$ it was shown by Lane and Wall [Trans. Amer. Math. Soc. **67** (1949), 368–380; MR **11**, 244] that the absolute convergence of $\{f_{2n-1}\}$ and $\{f_{2n}\}$ plus the divergence of $\sum |h_{2n-1}|$, where $h_n = 1/(1+a_{n+1}h_{n+1})$, implies the convergence of $f(a)$. For the continued fraction $g(b) = 1/b_1 + 1/b_2 + \cdots$ with approximants $\{g_n\}$, which is equivalent to $f(a)$ when $a_n = 1/b_n b_{n+1}$, the author shows that the absolute convergence of $\{g_{2n-1}\}$ plus the convergence of $\{g_{2N}\}$ and the divergence of $\sum |b_{2n-1}|$ implies the convergence of $g(b)$. An example is given to show that the theorem is not true if the condition $\sum |b_{2n-1}|$ divergent is replaced by the condition $\sum |b_n|$ divergent. It is also shown that if the inequalities of Scott and Wall [ibid. **47** (1940), 155–172; MR **1**, 217] of even index hold, if $a_{2n} \neq 0$ (so that $f_{2n-1} \to v$), and if

$$\sum |a_1 a_3 \cdots a_{2n-1}| / |a_2 a_4 \cdots a_{2n}|$$

diverges, then there is a subsequence of $\{f_{2n}\}$ which converges to v. This theorem is used to obtain results on convergence and absolute convergence of $f(a)$. (All of the above statements remain true if the roles of even and odd indices are interchanged.) *W. T. Scott* (Evanston, Ill.)

3694:

Midtdal, John. A practical method of summing convergent and semiconvergent series. Norske Vid. Selsk. Forh., Trondheim, **31** (1958), no. 17, 7 pp. (Norwegian)

A method is proposed for transforming a series into a more rapidly converging series. After n terms the series is presumed to continue as a geometric series with the ratio taken to be that of a_{n+1} to a_n of the given series. The sum of this hybrid infinite series is taken as the nth partial sum of the transformed series. This process can then be applied to this series to obtain still another transformed series, etc. Several numerical examples are worked out, including an asymptotic series, although no justification for the method is given for such series.

R. G. Langebartel (Urbana, Ill.)

3695:

Dikshit, G. D. On the absolute Riesz summability factors of infinite series. I. Indian J. Math. **1**, no. 1, 33–40 (1958).

The author proves the following extension of a result by U. Guha [J. London Math. Soc. **31** (1956), 311–319; MR **19**, 135] which concerns the convergence factors for Riesz summability.

Let $\sum a_n$ be summable $|R, \lambda, k|$ for some positive integer k; then $\sum a_n \cdot \psi(\lambda_n)$ will be summable $|R, \varphi(\lambda), k|$, if there is a positive function $f(x)$ such that (i) $x^{-1} \cdot f(x)$ is of bounded variation in (λ_1, ∞); (ii) $x^n \cdot \psi^{(n)}(x) = O[x^{n-k} \cdot f^{k-n}(x)]$ and (iii) $f^n(x) \cdot \varphi^{(n)}(x) = O[\varphi(x)]$ for $n = 1, 2, 3, \cdots, k$ and $x \geq \lambda_1$.

E. Kogbetliantz (New York, N.Y.)

3696:

Schoenberg, I. J. The integrability of certain functions and related summability methods. Amer. Math. Monthly **66** (1959), 361–375.

Let γ_n, $1 \leq n < \infty$, be any sequence. Put

$$s_n = \sum_{d|n} \varphi(d)\gamma_d \Big/ \sum_{d|n} \varphi(d),$$

where $\varphi(n)$ is Euler's φ-function. The author shows that $\gamma_n \to \lambda$ implies $s_n \to \lambda$, but that the converse is not true. In fact, he proves that $\lim s_n = \lambda$ implies $\gamma_{n_\nu} \to \lambda$ for every sequence γ_n if and only if

$$\liminf_{\nu \to \infty} \varphi(n_\nu)/n_\nu > 0.$$

Several other theorems are proved and questions of Riemann integrability of functions are discussed.

P. Erdős (Boulder, Colo.)

APPROXIMATIONS AND EXPANSIONS

See also 3585, 3718, 3719, 3723, 3730, 3737.

3697:

Stojaković, Mirko. Sur l'interpolation par polynomes à plusieurs variables. C. R. Acad. Sci. Paris **248** (1959), 3091–3093.

Following up earlier work [cf. same C. R. **246** (1958), 1133–1135; MR **19**, 1079], the author derives an expression for the interpolation polynomial in several variables from a certain matrix inversion formula for generalized Vandermonde matrices.

H. Schwerdtfeger (Montreal, P.Q.)

3698:

★Busk, Th. On a simple relationship between some of the classical interpolation formulae. Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18–23 août 1957, pp. 65–68. Mercators Tryckeri, Helsinki, 1958. 209 pp. (1 plate)

The author shows that Bessel's interpolation formula may be deduced from Stirling's by the formation of a (central) difference followed by a change of origin (x becomes $x+\frac{1}{2}$).

He also shows that Everett's and Steffenson's interpolation formulae are related in precisely the same way.

S. Macintyre (Cincinnati, Ohio)

3699a:

Saxena, R. B.; and Sharma, A. Convergence of interpolatory polynomials. Acta Math. Acad. Sci. Hungar. **10** (1959), 157–175. (Russian summary, unbound insert)

3699b:

Saxena, R. B. On modified (0, 2)-interpolation. Acta Math. Acad. Sci. Hungar. **10** (1959), 177–192. (Russian summary, unbound insert)

a. Let n be even. Based on the explicit formulae of a previous paper (*) [same Acta **9** (1958), 345–358; MR **21** #2138], the authors investigate the convergence of the polynomials $R_n(x)$ of degree $3n-1$ for which $R_n(x_\nu) = f(x_\nu)$, $R_n'(x_\nu) = f'(x_\nu)$ and moreover $\lim_{n\to\infty} \max_\nu |R_n''(x_\nu)| \cdot n^{-2} = 0$. If $f''(x)$ is continuous with modulus of continuity $\omega(\delta)$ and $\int_0 \omega(t)/t \cdot dt$ exists, the convergence takes place uniformly in $-1 \leq x \leq 1$.

b. This paper follows the general line of (*). The conditions imposed on $R(x)$ (of degree $2n+1$) are as follows: $R(x_\nu)$, $R''(x_\nu)$ are given, and also $R''(\pm 1)$. Again for n odd, x_ν symmetric, the solution is not unique. If n is even and the x_ν are the zeros of $(1-x^2)P'_{n-1}(x)$, the solution is unique and the fundamental polynomials are evaluated in explicit terms.

G. Szegő (Stanford, Calif.)

3700:

Stancu, Dumitru D. Sur une classe de polynômes orthogonaux et sur des formules générales de quadrature à nombre minimum de termes. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1** (**49**) (1957), 479–498.

The theory is here elaborated which gives a method for the construction of general quadrature formulas with a minimum number of terms. A class of symmetric orthogonal polynomials is studied and these are used for the establishment of formulas which generalize the classical quadrature formulas of Gauss. *E. Frank* (Chicago, Ill.)

3701:

Vzorova, A. I. Solution of the Laplace equation in a region interior to an ellipsoid. Vyčisl. Mat. **3** (1958), 88–98. (Russian)

La méthode est la suivante: les sommes partielles du développement $\sum a_{m,n}u_{m,n}$ d'une fonction f donnée continue sur un ellipsoïde E en série de polynômes harmoniques $u_{m,n}$ (convergentes à l'intérieur de E d'après Vekua [Dokl. Akad. Nauk SSSR **90** (1953), 715–718; MR **15**, 217]) sont identifiées aux sommes partielles d'un développement $\sum b_{m,n}v_{m,n}$ de f en série de polynômes $v_{m,n}$, orthogonaux sur E par rapport à une mesure bien choisie. Un exemple de calcul explicite est donné.

J. P. Kahane (Montpellier)

3702:

Campbell, Robert. Sur une extension des polynomes de Charlier et de Tschebycheff. C. R. Acad. Sci. Paris **248** (1959), 2937–2939.

The author considers the expansion in terms of a given set of orthogonal polynomials. The partial sums appear, as well known, in a closed form by the Christoffel-Darboux formula. The Cesàro sums of the first order can also be represented in a similar closed form for a certain class of special distributions. This class includes the Charlier-Poisson, Tchebychev, and Hermite distributions but not those of Laguerre. *G. Szegö* (Stanford, Calif.)

3703:

Golinskiĭ, B. L. Summation of Fourier-Čebysev series by the Fejér method. Mat. Sb. (N.S.) **47** (**89**) (1959), 255–264. (Russian)

Let f be a function of period 2π such that

$$\int |f(\phi)|^r p(\phi)d\varphi < \infty$$

and let $P_n(e^{i\phi})$ be polynomials orthonormal on $(0, 2\pi)$ with weight $p(\phi)$; let S_ν be the partial sums of the Fourier series of f with respect to the P_n, and let

$$\rho_n^{(k)}(\theta) = \left\{\frac{1}{n+1}\sum_{\nu=0}^{n}|S_\nu(\theta)-f(\theta)|^k\right\}^{1/k}.$$

The author proves the following theorems. If $r=2$ and $p(\phi)\geqq m>0$ almost everywhere in $[a-\eta, b+\eta]$, then $\rho_n^{(k)}(\theta)=o(1)$ for $k=1, 2$, almost everywhere in $[a, b]$. If also f is continuous in $[a, b]$ and $p(\phi)\leqq M$ almost everywhere in $[a, b]$ then $\rho_n^{(1)}(\theta)=o(1)$ uniformly in $[a, b]$. If $r=2$, $p(\phi)\geqq m>0$ almost everywhere in $[a-\eta, b+\eta]$ and is of bounded variation in $[0, 2\pi]$, then $\rho_n^{(k)}(\theta)=o(1)$ almost everywhere in $[a, b]$ for $k>0$. If in addition f is continuous in $[a, b]$ the conclusion holds uniformly.

R. P. Boas, Jr. (Evanston, Ill.)

3704:

Yahnin, B. M. Lebesgue functions for expansions in series of Jacobi polynomials for the cases

$$\alpha=\beta=\tfrac{1}{2},\quad \alpha=\beta=-\tfrac{1}{2},\quad \alpha=\tfrac{1}{2},\quad \beta=-\tfrac{1}{2}.$$

Uspehi Mat. Nauk **13** (1958), no. 6 (84), 207–211. (Russian)

The Lebesgue function $L_n^{(\alpha,\beta)}(x)$ for the Jacobi polynomials $J_k^{(\alpha,\beta)}(x)$ is defined as the sup taken over all continuous functions $f(x)$ for which $|f(x)|\leqq 1$ of the nth partial sum for the Fourier-Jacobi expansion of $f(x)$. The author uses $-1\leqq x\leqq 1$ for the domain of $f(x)$ instead of the perhaps more common $0\leqq x\leqq 1$. From the fact that $J_k^{(1/2,1/2)}(x)$ are the Chebishev polynomials so that for this case the expansion is in effect a Fourier expansion, several recursion type relations are derived, of which an example is

$$L_n^{(1/2,-1/2)}(x)= L_n^{(-1/2,-1/2)}(x)+O[1+|\sin(n+\tfrac{1}{2})\arccos x|(1-x)^{-1/2}] \quad (n\to\infty).$$

R. G. Langebartel (Urbana, Ill.)

3705:

Tonyan, V. A. Weighted polynomial approximation on the real axis. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **11** (1958), no. 4, 79–93. (Russian. Armenian summary)

The author gives proofs of theorems announced by him earlier [Dokl. Akad. Nauk SSSR **105** (1955), 656–658; MR **17**, 728]. He continues with further estimates for the degree of weighted uniform approximation by polynomials

$$E_n(f) = \inf_{P_n}\Big[\sup_{-\infty<x<+\infty} h(x)|f(x)-P_n(x)|\Big]$$

in the case when the weight $h(x)$ is logarithmically concave and $f(x)$ is either analytic in a strip $|\mathrm{Im}\, z|\leqq a$ or vanishes outside $|x|\leqq R$. *G. G. Lorentz* (Syracuse, N.Y.)

3706:

Garkavi, A. L. Dimensionality of polyhedra of best approximation for differentiable functions. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 93–114. (Russian)

Let (*) $\varphi_1(x), \cdots, \varphi_n(x)$ denote continuous functions on $[a, b]$, $P(x)=\sum_{k=1}^n a_k\varphi_k(x)$ polynomials in φ, and r a fixed integer with $1\leqq r\leqq n$. From results of G. Š. Rubinšteĭn [Dokl. Akad. Nauk SSSR **102** (1955), 451–454; MR **17**, 185] one derives necessary and sufficient conditions for the system (*) in order that each function $f\in C[a, b]$ should have at most r linearly independent polynomials of best approximation. Here a similar question is answered for the case when all φ and all approximated functions f belong to the space C_s of functions with s continuous derivatives, $s=1, 2, \cdots$. Let x_0 be called a special root of $P(x)$ if either $P'(x_0)=0$ or $x_0=a$, or $x_0=b$. The necessary and sufficient condition is independent of s and is the following: If $P_1, \cdots, P_k$, $r<k\leqq n$, are linearly independent polynomials, then among the points which are their common roots, there should be not more than $n-k$ special roots common to any $r+1$ of the polynomials $P_1, \cdots, P_k$. A similar simple statement is given in the case when the approximating polynomials P are restricted by a linear relation $\sum_1^n \alpha_k a_k=d$ for the coefficients of the P, and the system (*) possesses a certain regularity property. *G. G. Lorentz* (Syracuse, N.Y.)

3707a:

Berg, Lothar. Herleitung asymptotischer Ausdrücke für Integrale und Reihen. Wiss. Z. Hochschule Elektrotechn. Ilmenau **3** (1957), 5–7.

3707b:

Berg, Lothar. Hinreichende Bedingungen zur asymptotischen Darstellung von Parameterintegralen. Wiss. Z. Hochschule Elektrotechn. Ilmenau **3** (1957), 85–88.

The first paper is an abstract of the author's Habilitationsschrift, a revised version of which appeared in Math. Nachr. **17** (1958), 101–135 [MR **20** #5390]. The second paper contains several special results which are mentioned in the paper cited above.

R. P. Boas, Jr. (Evanston, Ill.)

FOURIER ANALYSIS

See also 3703.

3708:

Lauwerier, H. A. On certain trigonometrical expansions. J. Math. Mech. 8 (1959), 419–432.

In connection with a partial differential boundary problem, the author is led to study the expansion $f(x)=\sum_1^\infty b_k \sin(kx+\mu\pi)$ for x, $\mu\in[0,\pi)$ and μ constant. He finds that any f which is orthogonal to $(\tan x/2)^{-2\mu}$ (which in turn is orthogonal to all $\{\sin(kx+\mu\pi)\}_1^\infty$) has a unique such expansion, and he finds asymptotic expressions for the coefficients. The problem is generalized in two directions: (i) to complex μ's; and (ii) in place of $\{\sin(kx+\mu\pi)\}_1^\infty$, the set $\{\sin kx+\theta_k \cos kx\}_{k=1}^\infty$ is taken, where $\theta_k=\theta+O(\theta^{-c})$, $c>1$, for large k.

František Wolf (Berkeley, Calif.)

3709:

Fekete, Michael. On certain classes of periodic functions and their Fourier-series. Bull. Res. Council Israel. Sect. F 7 (1957/58), 103–112.

Let P denote the class of odd functions of period 2π which are Lebesgue integrable and non-negative on $0\leq x\leq\pi$. Let C be the subclass consisting of the convex functions in P. A necessary and sufficient condition that a function belong to either class P or C is that the Riesz mean of order two of its Fourier series belong to the same class. Let $\{\lambda_n\}$ be a sequence of real numbers. A necessary and sufficient condition that $\sum\lambda_n a_n \sin nx$ be the Fourier series of a function of class P or C whenever $\sum a_n \sin nx$ is the Fourier series of the same class is that $\sum n^{-1}\lambda_n \sin nx$ is the Fourier series of a function in C.

P. Civin (Eugene, Ore.)

3710:

Jakimovski, Amnon. Some remarks concerning "On certain classes of periodic functions and their Fourier-series" of M. Fekete. Bull. Res. Council Israel. Sect. F 7 (1957/58), 113–116.

The author considers classes of functions M_n^+ and M_n^- which are indefinite integrals of the elements in the corresponding class for $n-1$, and with M_0^+ and M_0^- are the positive and negative integrable functions on an interval. The author obtains results analogous to the final item quoted in the previous review. *P. Civin* (Eugene, Ore.)

3711:

Goes, Günther. *BK*-Räume und Matrixtransformationen für Fourierkoeffizienten. Math. Z. **70** (1958/59), 345–371.

The author considers several linear subspaces of the linear space P_∞ of all formal trigonometrical series

$$(a_n, b_n) \equiv \sum_1^\infty (a_n \cos nx + b_n \sin nx);$$

the most conspicuous are L_p $(1\leq p\leq\infty)$, C, V, A (the series is then the Fourier series of an L_p-function, a continuous function, a function of bounded variation, and an absolutely continuous function respectively). These subspaces are Banach spaces with respect to the usual norms; A is considered as a Banach subspace of the Banach space V. It is to be observed here that, e.g., for L_p, the space thus introduced is a subspace of what is usually called the L_p space over $[0, 2\pi]$, namely the subspace of all f for which the integral of f over $[0, 2\pi]$ is zero. Any linear subspace E of P_∞ in which a norm is defined such that E is complete with respect to this norm, and all $a_n=a_n(f)$, $b_n=b_n(f)$ are bounded linear functionals with respect to this norm, is an example of a BK-space; L_p, C, V and A are such BK-spaces. The following definitions are introduced (E is always a linear subspace of P_∞): The space $dE\subset P_\infty$ is the collection of all series such that the termwise integrated series belongs to E. If E is normed, then the norm in dE is, by definition, the same as the norm of the integrated series in E. The complementary space E^* of E is the collection of all (a_n, b_n) such that $\sum_1^\infty (a_nc_n+b_nd_n)$ converges for all $(c_n, d_n)\in E$. If E is a BK-space, the space E_N is the subspace of E consisting of those f for which the partial sums $s_n(f)$ of the Fourier series of f converge in norm to f. Finally, for $E=L_p$, C, V, A, the space E^0 is by definition L_q $(p^{-1}+q^{-1}=1)$, dV, dL_∞, dL_∞ respectively. Furthermore, if (a_n, b_n) and (c_n, d_n) are the Fourier series of f and g respectively, then $\sum_1^\infty (a_nc_n+b_nd_n)$ is denoted by $\langle f, g\rangle$.

The first main theorems are as follows: If E is a BK-space, then E^* is a BK-space with respect to the norm

$$\|f\|_{E^*} = \pi \sup_n \sup_{\|g\|_E=1} |\langle f, s_n(g)\rangle|.$$

If $E=L_p$, C, V, A, and $f\in E^*$, then

$$\|f\|_{E^*} = \sup_n \|s_n(f)\|_{E^0},$$

and $(E^0)_N\subset E^*\subset E^0$. In particular, for $E=L_p$ $(1\leq p<\infty)$, C, A we have $f\in E^*$ if and only if $\sup\|s_n(f)\|_{E^0}<\infty$. For $E=L_p$ $(1<p<\infty)$ it is well known that $(E^0)_N=E^0=L_q$, but for $E=L_1$, C, V, A it is shown here that all inclusions are proper.

Next, if the infinite matrix (a_{ij}) $(i, j=1, 2, \cdots)$ is given, the trigonometrical series (a_n, b_n) is said to be transformed into the trigonometrical series (a_n', b_n') by the matrix (a_{ij}) if $a_n'=\sum_j a_{nj}a_j$ and $b_n'=\sum_j a_{nj}b_j$. Notation: if $f=(a_n, b_n)$, then $Tf=(a_n', b_n')$ and

$$t_nf = \sum_1^n (a_i' \cos ix + b_i' \sin ix).$$

Of course, the transformation T can be applied to (a_n, b_n) if and only if the series for a_n' and b_n' converge. We write $T\in(E, E_1)$ if every series in E is transformed into a series in E_1. Hence, $T\in(E, P_\infty)$ means simply that T may be applied to every series in E. A number of necessary and

sufficient conditions for $T \in (E, E_1)$ are derived for combinations (E, E_1) of L_p, C, V, A. The norms $\|t_n\|_{EE_1} = \sup \|t_n f\|_{E_1}$ for $\|f\|_E \leq 1$ play the role of Lebesgue constants. One example is: If E is a Banach space containing all trigonometric polynomials, and E_1 is one of the spaces L_p $(1 \leq p < \infty)$, C, A, then $T \in (E, E_1{}^*)$ if and only if $\|t_n\|_{EE_1{}^0}$ is bounded as $n \to \infty$. Finally, in the last part of the paper, conditions of the form: $T \in (E, E_1)$ implies $T^* \in (E_1{}^*, E^*)$ or $T^* \in (E_1{}^*, E^0)$ are derived. Here, T^* is the transformation corresponding to the transposed matrix $(a_{ij}{}^*) = (a_{ji})$. Several results in the paper extend earlier results of F. H. Young [Proc. Amer. Math. Soc. **3** (1952), 783–791; MR **14**, 464]. At the end, the author offers apologies to the classical analysts for his use of methods of functional analysis, and to the functional analysts for a number of theorems which are merely valid for special choice of E and E_1. *A. C. Zaanen* (Leiden)

3712:

Menchoff, D. Sur les suites convergentes des sommes partielles des séries trigonométriques. Colloq. Math. **6** (1958), 155–164.

A summary of results obtained by the author in previous papers [Trudy Mat. Inst. Steklov. **32** (1950); Dokl. Akad. Nauk SSSR **114** (1957), 476–478; MR **12**, 254; **20** #1156] and in a paper to appear in Mat. Sbornik. No proofs are given. *M. Cotlar* (Buenos Aires)

3713:

Chow, Hung Ching. An additional note on the strong summability of Fourier series. J. London Math. Soc. **33** (1958), 425–435.

Continuing his previous studies [Proc. London Math. Soc. (3) **1** (1951), 206–216; Ann. Acad. Sinica. Taipei **1** (1954), 559–567; MR **13**, 340; **16**, 1099] the author obtains further results on the strong summability of Fourier series

$$(*) \qquad \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} c_n(x)$$

of functions $f(t) \in L(0, 2\pi)$ with period 2π.

We quote the following: Let $1 < p \leq 2$, $p' = p/(p-1)$ and $-1/p' < \alpha < 0$. If

$$\sum_{n=1}^{m} |n^{-\alpha} c_n(x)|^{p'} = O(m),$$

$$\int_0^t |f(x+u) + f(x-u)|^p du = O(t) \quad \text{as} \quad t \to +0,$$

then

$$\sum_{n=1}^{m} |s_n{}^\alpha(x)|^{p'} = O(m),$$

where $s_n{}^\alpha(x)$ denote the (C, α) means of the sequence of partial sums of $(*)$. *A. Dvoretzky* (Jerusalem)

3714:

Flett, T. M. A local property of Fourier series. J. London Math. Soc. **33** (1958), 450–454.

Let

$$f(\theta) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) = \sum_{n=0}^{\infty} A_n(\theta)$$

and let $\sigma_n{}^\alpha = \sigma_n{}^\alpha(\theta)$ be the nth Cesàro mean of order α of this series. The series is said to be strongly summable $(C, \alpha+1)$ with index k to the sum s (here $k \geq 1$, $\alpha > -1$) if $\sum_0{}^m |\sigma_n{}^\alpha - s|^k = o(m)$. This property is denoted here $\{C, \alpha\}_k$ (the usual notation is $[C, \alpha+1]_k$). The author proves that within the family L^p $(p \geq 1)$, $\{C, \alpha\}_k$ is a local property if and only if $\alpha \geq \sup(1/p-1, -1/k, -1/2)$. This contains as special cases results of Hardy and Littlewood [Proc. London Math. Soc. (2) **26** (1927), 273–286] $(\alpha = 0, p = 1)$ and of Chow [see preceding review and references] $(1 < p \leq 2$, $k = p/(p-1)$ and $p = 1$, $\alpha < 0)$. *A. Dvoretzky* (Jerusalem)

3715:

Gupta, D. P. On the Cesàro summability of the ultraspherical series (2). Proc. Nat. Inst. Sci. India. Part A **24** (1958), 419–440.

Any result obtained for the trigonometric or Laplace series is susceptible of extension to the expansion on the sphere (θ, φ) of a $f(\theta, \varphi)$ in its ultraspherical series, since this series includes, for particular values of its parameter λ, the Laplace series $(\lambda = \frac{1}{2})$ as well as (for $\lambda \to 0$) the Fourier trigonometric series. But the proof of such an extension for any λ is a much more difficult task.

In 1928 Hardy and Littlewood found that the trigonometric series of $f(x)$ is summable (C, k) for $-1 < k < 0$, provided that $f(x)$ satisfies the Lipschitz condition Lip* $(-k)$ of order $-k$. The corresponding theorem for the Laplace series was obtained by Du-Plessis [J. London Math. Soc. **27** (1952), 337–352; MR **13**, 935]: the Laplace series is summable (C, k) for $-\frac{1}{2} < k < \frac{1}{2}$ if the spherical mean of $f(\theta, \varphi)$ satisfies the Lipschitz condition of the order $\frac{1}{2} - k$.

After a considerable and very detailed analysis the author succeeded in extending these theorems to ultraspherical series for any λ in the interval $0 < \lambda \leq \frac{1}{2}$: (C, k) summability for $\lambda - 1 < k < \lambda$ is a corollary of the hypothesis that the "λ-generalized" spherical mean of $f(\theta, \varphi)$ satisfies the Lipschitz condition of order $\lambda - k$. {In the opinion of the reviewer this result can, and sooner or later will, be extended to the whole range of the parameter λ, but this task is not an easy one.} *E. Kogbetliantz* (New York, N.Y.)

3716:

Sharma, P. L. On the harmonic summability of double Fourier series. Proc. Amer. Math. Soc. **9** (1958), 979–986.

Eine Doppelfolge $(s_{m,n})$ heisst $(H, 1, 1)$-summierbar, wenn

$$\lim_{n,m \to \infty} \frac{1}{\log m \log n} \sum_{l=0}^{m} \sum_{k=0}^{n} \frac{s_{m-l}s_{n-k}}{(l+1)(k+1)}$$

existiert. Der Verf. zeigt, dass die Fourierreihe einer Funktion $f(u, v)$ an jedem Punkt $(H, 1, 1)$-summierbar ist, für den gilt

$$\int_0^u ds \int_0^v |\phi(s, t)| dt = o(uv(\log u^{-1})^{-1}(\log v^{-1})^{-1}),$$

$$\int_0^v dt \left| \int_0^u \phi(s, t) ds \right| = O(u(\log u^{-1})^{-1}),$$

$$\int_0^u ds \left| \int_0^v \phi(s, t) dt \right| = O(v(\log v^{-1})^{-1})$$

($0 \leq u, v \leq \pi$, $u, v \to 0$), wobei

$$\phi(s, t) = \tfrac{1}{4}[f(x+s, y+t)+f(x+s, y-t)+f(x-s, y+t)+f(x-s, y-t)-4s]$$

gesetzt sei. Hierdurch wird ein Ergebnis von Hille und Tamarkin [Trans. Amer. Math. Soc. **34** (1932), 757–783] verallgemeinert [vgl. auch J. A. Siddiqi, Proc. Indian Acad. Sci. Sect. A **28** (1948), 527–531; MR **10**, 369].

A. Peyerimhoff (Giessen)

3717:

★**Nikolsky, S. M. Einige Fragen der Approximation von Funktionen durch Polynome.** Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, Vol. 1, pp. 371–382. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam; 1957. 582 pp. $7.00.

The author gives a useful, though concentrated, resumé of key problems and results in two fields, to both of which he has himself contributed notably. The greater part of the paper is devoted to approximation by trigonometric polynomials to functions of one variable with various differentiability properties, with a brief reference to approximation by entire functions. The other main topic is that of inter-relations between the differentiability properties of functions of several variables, with an application to the Dirichlet problem, the role of approximation theory in this field being briefly indicated.

F. V. Atkinson (Canberra)

3718:

Frey, Tamás. On local best approximation by polynomials. I. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **7** (1957), 403–412. (Hungarian)

Let $\{S_n(t)\}$ be the sequence of Dirichlet kernels of Fourier series, $E_n(t)$ the nth order Euler mean of the former and $F_n(t) = n^{-1}\sum_{l=n}^{2n-1} E_l(t)$. The following properties of F_n are proved: (a) $F_n - S_n$ is orthogonal to every trigonometric polynomial of order n; (b) $\int_{-\pi}^{+\pi} |F_n(t)|dt < 17$; (c) $\int_{\delta}^{\pi} |K_n(t)|dt < C_1(\delta)n^{-1}q^n$ with $q = \cos \delta/2 < 1$. Let us put $A_n(f; x) = f * F$ and suppose that for some trigonometric polynomial T_n of order n we have $|f(x) - T_n(x)| \leq \varepsilon_n^{(1)}$ for $x \in [x_0 - \delta, x_0 + \delta]$, and $|f(x) - T_n(x)| \leq \varepsilon_n^{(2)}$ for arbitrary x. It follows that

$$|f(x_0) - A_n(f; x_0)| \leq 18\varepsilon_n^{(1)} + C_2(\delta)n^{-1}q^n\varepsilon_n^{(2)}.$$

This result improves a former theorem of Bochner which was stated to be best possible [S. Bochner, Ann. of Math. (2) **52** (1950), 642–649; MR **12**, 283]. The mistake in Bochner's argument is attributed to an erroneous use of the Poisson-Jensen inequality.

G. Freud (Rome)

3719:

Frey, Tamás. On local best approximation by polynomials. II. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **8** (1958), 89–112. (Hungarian)

With the same notation as in the preceding review, under the single condition that $|f(x) - T_n(x)| \leq \varepsilon_n$, $x \in [x_0 - \delta, x_0 + \delta]$, we have

$$|A_n(f; x_0) - f(x_0)| \leq C_3(\delta)\varepsilon_n + C_4(\delta)q_1^n,$$

with some $q_1 = q_1(\delta) < 1$. An exposition follows about local structure of functions with prescribed local approximation properties, along the line of a paper of S. B. Stečkin [Izv. Akad. Nauk SSSR **15** (1951), 219–242; MR **13**, 29]. {Reviewer's remark: Lemma 31.2, which is not used later, seems to be false.}

G. Freud (Rome)

3720:

Efimov, A. V. Approximation of conjugate functions by Fejér sums. Uspehi Mat. Nauk **14** (1959), no. 1 (85), 183–188. (Russian)

Soit f une fonction continue 2π-périodique, nulle en 0, dont le module de continuité d'ordre k ($k \geq 2$) [pour la définition, voir par exemple l'analyse de Bari et Stečkin, MR **18**, 303] satisfait $\omega_k(h) \leq h$. Soient $\tilde{f}$ et $\tilde{\sigma}_n$ les fonctions conjuguées de f et de sa somme de Fejér d'ordre n. Théorème 1:

$$\tilde{f}(x) - \tilde{\sigma}_n(x) = f\left(x - \frac{1}{2n}\right) - f\left(x + \frac{1}{2n}\right) + O(1/n).$$

Théorème 2:

$$f(y)/y - f(z)/z = O(1 + \log(z/y)).$$

J. P. Kahane (Montpellier)

3721:

Hirschman, I. I., Jr. On multiplier transformations. Duke Math. J. **26** (1959), 221–242.

Let $\mathfrak{J}$ denote the integer group, while $\mathfrak{R}$ denotes the additive group of the reals and $\mathfrak{T} = \mathfrak{R}/\mathfrak{J}$ is the circle group. Let G be any of these groups, and $\hat{G}$ its character group. The Fourier transformation $Y(G, \pm)$ is the continuous linear extension to $L^2(\hat{G})$ of the bounded transformation defined for all step-functions f by the relation

$$[Y(G, \pm)f](y) = \int_G f(x) \cdot \exp(\pm 2\pi ixy) \cdot dx$$

(Haar integral, for all y in $\hat{G}$).

Set $Y_- = Y(G, -)$ and $Y_+ = Y(\hat{G}, +)$. If $g \in L^\infty(G)$, then $[Tg](a) = Y_-(g \cdot Y_+ a)$, and Tg is the mapping $a \to [Tg](a)$ defined for all a in $L^2(\hat{G}) \cap L^p(\hat{G})$; the author calls Tg the "multiplier transformation determined by" g; it is said to be "bounded on" $L^p(\hat{G})$ if it admits a continuous linear extension to the whole space $L^p(\hat{G})$. More precisely, Tg is "bounded on" $L^p(\hat{G})$ if there exists a number $A > 0$ such that $A \geq \|[Tg](a)\|_p/\|a\|_p$ whenever $a \neq \mathbf{0}$ and $a \in L^2(\hat{G}) \cap L^p(\hat{G})$; here $f \to \|f\|_p$ is the usual norm of the space $L^p(\hat{G})$. The central purpose of this article is to supply conditions on g which will insure that Tg is bounded on $L^p(\hat{G})$. The condition $g \in L^\infty(G)$ is assumed throughout. Some of the typical results will now be described in the case where $\hat{G} = \mathfrak{J}$; these involve the following condition: there exists a number $B > 0$ such that $B \geq \sum_k |g(\theta_{k+1}) - g(\theta_k)|^\beta$ ($k = 0, 1, \cdots, n-1$) for all finite sequences $\{\theta_k : k = 0, 1, \cdots, n-1\}$ in G having

$$\theta_0 < \theta_1 < \cdots < \theta_{n-1} < \theta_n = \theta_0 + 1.$$

Theorem 2f: if $\beta \geq 1$ then Tg is bounded on $L^p(\hat{G})$ for all values of p in the open interval $I_\beta = (2\beta/(\beta+1), 2\beta/(\beta-1))$. Since $I_1 = (1, \infty)$, this generalizes a theorem of Stečkin [Dokl. Akad. Nauk SSSR **71** (1950), 237–240; MR **11**, 504]. Theorem 2e requires that $\beta \geq 2$ and concludes that Tg is bounded on $L^p(\hat{G})$ whenever $2\beta/(\beta+2) < p < 2\beta/(\beta-2)$, provided there exists a $\delta > 0$ and an $M > 0$ such

that the relation $|g(\theta)-g(h)| \leq M|\theta-h|^\beta$ holds for all pairs (θ, h) in $G\times G$.

Several other interesting results are proved; some of the more intricate have analogues in the case $G=\mathfrak{R}^n$ and require the differentiability of g. Suppose $G=\mathfrak{R}^n$: the author formulates without proof an analogue of Theorem 2e; he also states that Theorem 2f has an analogue, but (to this reviewer's regret) no proof is given. The article deals also with the eventuality $G=\mathfrak{Z}$, and the dual of Theorem 2e is proved. *G. L. Krabbe* (Lafayette, Ind.)

3722:

Devinatz, A.; and Hirschman, I. I., Jr. Multiplier transformations on $l^{2,\alpha}$. Ann. of Math. (2) **69** (1959), 575–587.

Let $\mathfrak{S}'$ be the set of all complex-valued functions on $\Omega=\{0, \pm 1, \pm 2, \cdots\}$. The set $l^{2,\alpha}$ consists of all F in $\mathfrak{S}'$ such that $\infty \neq N_\alpha(F)=\sum\{|F(n)|^2(|n|+1)^{2\alpha};\ n\in\Omega\}$; the norm $F\rightarrow\|F\|_{2,\alpha}=[N_\alpha(F)]^{1/2}$ makes $l^{2,\alpha}$ into a Banach space. This article deals with linear transformations of $l^{2,\alpha}$ that commute with convolution.

Let $\mathfrak{S}$ be the set of functions in $\mathfrak{S}'$ having compact support. Let $\mathfrak{M}$ consist of all linear mappings T of $\mathfrak{S}$ into $\mathfrak{S}'$ such that $T(F*G)=(TF)*G$ whenever $(F, G)\in\mathfrak{S}\times\mathfrak{S}$; note that $F*G$ is the convolution $n\rightarrow\sum\{F(n-k)\cdot G(k);\ k\in\Omega\}$. Let $D(n)=1$ if $n=0$, while $D(n)=0$ whenever $n\neq 0$ $(n\in\Omega)$. Suppose $-1/2<\alpha<1/2$ and $T\in\mathfrak{M}$ throughout. It is easily seen that T is the convolution operator $F\rightarrow(TD)*F$ defined on $\mathfrak{S}$. As usual, the operator-norm $\mathfrak{N}_\alpha[T]$ is the supremum of the set $\{\|TF\|_{2,\alpha}:\ F\in\mathfrak{S}$ and $\|F\|_{2,\alpha}\leq 1\}$. Let $\mathfrak{M}_\alpha$ denote the set of all T in $\mathfrak{M}$ such that $\mathfrak{N}_\alpha[T]\neq\infty$. It turns out that $\mathfrak{M}_\alpha$ is a semi-simple, abelian Banach algebra. Suppose $T\in\mathfrak{M}_0$, and define $T^\sim$ as the Fourier transform of TD; then T is the "multiplier transformation determined by $T^\sim$" in the sense defined by Hirschman [review above].

Suppose $T\in\mathfrak{M}_\alpha$; it is proved that the spectrum $\sigma(T;\mathfrak{M}_\alpha)$ coincides with the essential range of the function $T^\sim$. Moreover, $\|T^\sim\|_\infty$ is the spectral norm of T (in the algebra $\mathfrak{M}_\alpha$). The proofs are based partly on a thesis of A. Broman [Uppsala, 1947; MR **9**, 182]; they also depend on articles contained in the Cornell Symposium on Harmonic Analysis (1956).

Unlike the spectrum, the point-spectrum $P(T;\mathfrak{M}_\alpha)$ varies with α. The authors denote by $A(\lambda)$ the set of all θ in the domain of $T^\sim$ such that $T^\sim(\theta)=\lambda$. The second part of the article generalizes the following result of Toeplitz: if $T\in\mathfrak{M}_0$, then a necessary and sufficient condition that $\lambda\notin P(T;\mathfrak{M}_0)$ is that $A(\lambda)$ be a μ_0-negligible set ($\mu_0=$ Lebesgue measure). In case $\alpha>0$ the authors define on the Borel sets a set-function μ_α (it is a "capacity" in the sense of Choquet). The Fourier transform $T^\sim$ of TD can accordingly be re-defined by the relation

$$T^\sim(\theta)=\sum\{[TD](n)\cdot\exp 2\pi in\theta;\ -\infty<n<\infty\}$$

almost-everywhere for the capacity μ_α. This comes from Broman's thesis [loc. cit.]. Results of C. S. Herz in the Cornell Symposium lead to the following theorem: if $\alpha>0$ and $T\in\mathfrak{M}_\alpha$, then a necessary and sufficient condition that $\lambda\notin P(T;\mathfrak{M}_{-\alpha})$ is that $A(\lambda)$ be a μ_α-negligible set. This still holds in the case $\alpha<0$, when μ_α is re-defined in a suitable way; in this case it is not a capacity.

G. L. Krabbe (Lafayette, Ind.)

3723:

Korevaar, Jacob. Pansions and the theory of Fourier transforms. Trans. Amer. Math. Soc. **91** (1959), 53–101.

L'auteur expose de façon nouvelle la théorie de la transformation de Fourier $(f\rightarrow Tf)$ fondée sur le développement de f en série de fonctions de Hermite (1) $\sum_0^\infty c_kv_k$. Les v_k sont définies comme fonctions propres, correspondant aux valeurs propres $l=2k+1$, du problème de l'oscillateur harmonique $Hy=x^2y-D^2y=ly$, $D=d/dx$, $\int_{-\infty}^\infty|y|^2dx=1$. Dans la classe des fonctions f telles que, pour un $a<\frac{1}{2}$, $f\exp(-ax^2)\in L^1$, les v_k forment un système complet; chaque f admet un développement (1) qui la caractérise, et on obtient facilement les développements de Dx, xf, Hf et $Tf(\sim\sum_0^\infty(-i)^kc_kv_k)$. Il arrive qu'une somme (1) ne soit, dans aucun sens naturel, un développement (expansion) de fonction; l'auteur l'appelle alors "pansion" (ce que le référent, à son regret, traduira par veloppement); on définit, pour des veloppements ϕ et ψ, les veloppements $D\phi$, $x\phi$, $H\phi$, $e^{D}\phi$ (opérateur de translation), $\phi*\psi$, $T\phi$. Certaines propriétés de ϕ sont liées de façon simple aux c_k: (1) $\{c_k\}\in l^2\Leftrightarrow\phi\in L^2$, d'où le théorème de Plancherel. (2) $\{c_k\}$ à croissance polynomiale $\Leftrightarrow\phi$ est un "veloppement à croissance polynomiale" ("distributions à croissance lente" de L. Schwartz) $\Leftrightarrow\phi\in$ plus petit espace vectoriel contenant L^2 et fermé pour les opérations $\phi\rightarrow D_\phi$ et $\phi\rightarrow x\phi$. (3) $\log|c_k|=O(\sqrt{k})\Leftrightarrow\phi$ est un "veloppement à croissance exponentielle" $\Leftrightarrow\phi\in$ plus petit espace vectoriel contenant L^2 et fermé pour les opérations $\phi\rightarrow E(D)\phi$ et $\phi\rightarrow E(x)\phi$, correspondant à toutes les fonctions E entières de type exponentiel $\Leftrightarrow\phi$ $(\cos aD)(\cos bx)f$, $a\geq 0$, $b\geq 0$, $f\in L^2$. Cet article peut être lu par un vaste public; il nécessite peu de connaissances préalables, indique les calculs en détail et constitue une bonne introduction à l'étude des distributions et distributions généralisées.

J. P. Kahane (Montpellier)

3724:

Ou, Šo-mo. Completeness of sets of functions. Acta Math. Sinica **7** (1957), 477–491. (Chinese. English summary)

English version reviewed below.

3725:

Ou, Šo-mo. Completeness of sets of functions. Sci. Sinica **7** (1958), 829–843.

The author proves a number of theorems on completeness of sets of functions in certain L_p and H_p spaces, usually be reducing the problem to a uniqueness theorem for analytic functions. We cite the following result. Let $g(x)$ be non-decreasing, left-continuous on $[-\pi,\pi]$ and such that $\int_{-\pi}^{\pi}|\log g'(x)|dx=\infty$. $L_p(g)$ denotes the L_p space on $[-\pi,\pi]$ with respect to the mass distribution dg. Let $0<b_1<b_2<\cdots\rightarrow\infty$, and let $b(r)$ denote the number of $|b_n|<r$. Theorem: $\{\exp(ib_nx)\}$ is complete in $L_p(dg)$ $(p>1)$ if either of the following conditions is satisfied:

(a) $$\lim\int_1^R b(r)(1/R^2+1/r^2)dr=\infty\quad(R\rightarrow\infty),$$

(b) $$\limsup_{a\rightarrow 1-}\ \limsup_{r\rightarrow\infty}\frac{b(r)-b(ar)}{r(1-a)}=A>1\quad(0<a<1).$$

A. Shields (Ann Arbor, Mich.)

3726a:

Albrycht, J. Some remarks on the Marinkiewicz-Orlicz space. II. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 11–12.

3726b:

Albrycht, J. Some remarks on the Marcinkiewicz-Orlicz space. III. Bull Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 55–56.

[For part I, see Bull. Acad. Polon. Sci. Cl. III **4** (1956), 1–3; MR **17**, 953.] The author discusses properties of new classes of almost periodic functions defined on $(-\infty, +\infty)$ or, in the second paper, on an arbitrary infinite group. His classes differ from the classes of Besicovitch and of Følner [Math. Scand. **5** (1957), 47–53; MR **19**, 958] in the sense that the L^p-metric is replaced by an Orlicz metric.

G. G. Lorentz (Syracuse, N.Y.)

3727:

Musielak, J. A note on Fourier series of functions of an infinite number of variables. Ann. Polon. Math. **6** (1959/60), 69–73.

If A and $\bar{A}$ are complementary sets of positive integers, if $m=(m_1, m_2, \cdots)$ is a sequence of non-negative integers with only finitely many different from 0, and if $n(m)$ is the number of positive integers in m, the author defines

$$\phi_m{}^A(x) = 2^{n(m)/2}\prod_{i\in A}\cos 2\pi m_i x_i \prod_{j\in\bar{A}}\sin 2\pi m_j x_j$$

where $x=(x_1, x_2, \cdots)$, $0\leq x_k<1$. The functions $\phi_m{}^A$ form a complete orthonormal system on the infinite-dimensional torus Q_ω; the Fourier coefficients of any $f\in L^2(Q_\omega)$ with respect to this system are denoted by $a_m{}^A(f)$. The paper establishes two sufficient conditions which guarantee the convergence of the series

$$\sum_{m,A} 2^{\alpha\gamma n(m)}|a_m{}^A(f)|^\gamma \quad (\alpha \text{ real},\ 0<\gamma<2).$$

These rather complicated conditions involve a certain integral modulus of continuity and a certain analogue of the total variation. *W. Rudin* (Madison, Wis.)

3728:

Rudin, Walter. Representation of functions by convolutions. J. Math. Mech. **7** (1958), 103–115.

Let $L^1(R^1)$ denote the class of complex-valued functions on the real line R^1 which are absolutely integrable with respect to Lebesgue measure, and for $f, g\in L^1(R^1)$ let $f*g$ be defined on the usual fashion. It is well known that $f*g\in L^1(R^1)$. In an earlier paper [Proc. Nat. Acad. U.S.A. **43** (1957), 339–340; MR **19**, 46], the author proved that every function h in $L^1(R^1)$ can be represented as a convolution $h=f*g$; that is, there are no primes in the convolution algebra $L^1(R^1)$. In the present paper, this result is extended to all abelian groups G which are locally Euclidean. The basic step in the argument is the deduction of the theorem for R^n. It is then carried over to T^n (the n-dimensional torus group) by summing over lattice points in R^n. The final step follows on noting that G is of the form $T^n\oplus R^k\oplus D$, where D is discrete.

{The author has pointed out, in a letter, that his results contradict footnote 3 of J. Dieudonné, Compositio Math. **12** (1954), 17–34 [MR **16**, 265]. He writes that Dieudonné agrees that his footnote is incorrect.}

I. I. Hirschman (St. Louis, Mo.)

3729:

Cohen, Paul J. Factorization in group algebras. Duke Math. J. **26** (1959), 199–205.

The author shows that any integrable function on a locally compact group is the convolution of two such functions. Rudin [article reviewed above] had shown this for Euclidean n-space and the n-torus. The author proves it in general by showing that in any Banach algebra with an approximate identity every element is a product. He also shows that a strictly positive continuous function on a compact group is the convolution of two strictly positive functions, but gives an example of a non-negative continuous function on the circle that is not the convolution of two non-negative functions.

W. F. Stinespring (Princeton, N.J.)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

See also 3728, 3761, 3762.

3730:

Nudel'man, A. A. A generalization of some results obtained by P. L. Čebyšev and A. A. Markov. Dokl. Akad. Nauk SSSR **125** (1959), 740–742. (Russian)

[For terminology and references see M. G. Krein, Uspehi Mat. Nauk (N.S.) **6** (1951), no. 4 (44), 3–120; MR **13**, 445.] It was shown by A. A. Markov for the interval $(0, \infty)$ and by P. G. Rehtman-Ol'šanskaya [ibid. **12** (1957), no. 3 (75), 181–187; MR **19**, 741] for a finite interval that if $\{a_k\}_0{}^n$ and $\{b_k\}_0{}^n$ are power moments $\int t^k d\sigma(t)$, $\sigma(t)$ non-decreasing, then the same is true for any $\{s_k\}_0{}^n$ with $(-1)^{k+1}a_k\leq(-1)^{k+1}s_k\leq(-1)^{k+1}b_k$. The author extends this to moments $\int u_k(t)d\sigma(t)$ under suitable hypotheses. The $u_k(t)$ are continuous and the determinant

$$\Delta\begin{pmatrix}u_0\, u_1 \cdots u_n\\ t_1\, t_2 \cdots t_{n+1}\end{pmatrix} = \begin{vmatrix}u_0(t_1) & u_1(t_1) & \cdots & u_n(t_1)\\ \cdot & \cdot & \cdots & \cdot\\ u_0(t_{n+1}) & \cdot & \cdots & u_n(t_{n+1})\end{vmatrix}$$

and all its n-rowed minors are positive for $a\leq t_1<\cdots<t_{n+1}\leq b$. Let K denote the convex cover of the set of curves $x_k=u_k(t)$, $a\leq t\leq b$; then K is the set of all points of $(n+1)$-space whose coordinates are moments. Theorem 1: If $A=(a_0, a_1, \cdots, a_n)$ and $B=(b_0, b_1, \cdots, b_n)$, where $(-1)^{k+1}a_k\leq(-1)^{k+1}b_k$, belong to K, then so do all points $S=(s_0, \cdots, s_n)$ such that

$$(-1)^{k+1}a_k \leq (-1)^{k+1}s_k \leq (-1)^{k+1}b_k.$$

Theorem 2: Let $\Omega(t)$ be such that

$$\Delta\begin{pmatrix}u_0\, u_1 \cdots u_n & \Omega\\ t_1\, t_2 \cdots t_{n+1} & t_{n+2}\end{pmatrix}$$

and all the minors of the elements $u_k(t_j)$ are positive for all t_k. Then the smallest [largest] value of $\int_a{}^b \Omega(t)d\sigma(t)$ for $(-1)^{k+1}a_k\leq(-1)^{k+1}\int_a{}^b u_k(t)d\sigma(t)\leq(-1)^{k+1}b_k$ and $A, B\in K$ is attained for the $\sigma(t)$ which is, respectively, for $n=2m-1$ the lower [upper] principal representation of $\{a_k\}_0{}^n$ [$\{b_k\}_0{}^n$]; for $n=2m$, interchange the expressions in brackets. Application: let f_1 and f_2 be absolutely monotonic on $(-\infty, 0]$, and $(-1)^{k+1}f_1(x_k)\leq(-1)^{k+1}f_2(x_k)$, $x_0<x_1<\cdots<x_n<0$. Then if $(-1)^{k+1}f_1(x_k)\leq(-1)^{k+1}y_k\leq(-1)^{k+1}f_2(x_k)$, there is an absolutely monotonic function f with $f(x_k)=y_k$. *R. P. Boas, Jr.* (Evanston, Ill.)

3731:

Saksena, K. M. On a generalized Laplace integral. Math. Z. 68 (1957), 267–271.

The differential operators U_q, V_q are defined by $U_0[f(x)]=f(x)$, $U_q[f(x)]=(-1)^q x^{m-k-q-1/2}D^q[x^{k-m+1/2}f(x)]$ $(q=1, 2, 3, \cdots)$, where $D\equiv x^2d/dx$, and

$$V_q[f(u)] = \frac{u^{-1}}{\Gamma(q+k+m-\frac{1}{2})}[U_q[f(x)]]_{x=u^{-1}(q+k+m)-1/2)}.$$

The generalized Laplace transform is taken in the form

$$(\alpha)\qquad f(x) = \int_0^\infty (xt)^{m-\frac{1}{2}}e^{-\frac{1}{2}xt}W_{k,m}(xt)\phi(t)dt,$$

where $W_{k,m}(x)$ is Whittaker's function.

The author proves an inversion formula for (α) which he has given previously [Proc. Nat. Inst. Sci. India 19 (1953), 173–181; MR 14, 1081; p. 179]. He then proves that the necessary and sufficient conditions that a given function $f(x)$ may have the representation (α) with $\phi(t)$ bounded are that (i) $f(x)$ is infinitely differentiable on $(0, \infty)$, (ii) $f(x)\to 0$ for $x\to\infty$, and (iii) $|V_q[f(u)]|\leq M$ $(0<u<\infty$, $q=1, 2, \cdots)$, where M is a constant.

J. L. Griffith (Kensington)

3732:

Narain, Roop. Certain properties of generalized Laplace transform involving Meijer's G-function. Math. Z. 68 (1957), 272–281.

Define

$$\phi(s: k, m) = W[f(t); k, m]$$

$$= s\int_0^\infty (st)^{m-\frac{1}{2}}e^{-\frac{1}{2}st}W_{k,m}(st)f(t)dt,$$

where $W_{k,m}(x)$ is Whittaker's function. Suppose that $W[t^\sigma h(t); \lambda, \mu]=\phi(s: \lambda, \mu: \sigma)$ and that $W[g(t); k, m]=s^\rho h(1/s)$; it is proved that

$$\text{(i)}\quad \int_0^\infty g(t)G_{2,4}^{1,0}\left(st\left|\begin{matrix}\frac{1}{2}-k+m,\ \nu-\lambda\\ 2m,\ 0,\ \nu+\mu-\frac{1}{2},\ \nu-\mu-\frac{1}{2}\end{matrix}\right.\right)dt = s^{\nu-\mu-3/2}\phi(s: \lambda, \mu: \nu-\mu-\rho-\tfrac{1}{2})$$

provided that $R(s)>0$, the integral in (i) is absolutely convergent and the generalised Laplace transforms (the W-transforms) of $|g(t)|$ and $|t^{\nu-\mu-\rho-1/2}h(t)|$ exist. The function denoted by $G_{2,4}^{1,0}(\cdot)$ is one of Meijer's G-functions.

Three similar theorems are given, together with a number of corollaries found by giving special values to the various parameters. *J. L. Griffith* (Kensington)

3733:

Mayer-Kalkschmidt, J. A theorem on Laplace-Stieltjes-integrals. Proc. Amer. Math. Soc. 10 (1959), 286–293.

The (C, k)-mean of the Laplace-Stieltjes integral

$$(*)\qquad \int_0^\infty e^{-st}da(t)$$

is defined by the function

$$m_k(s, t) = t^{-k}\int_0^t (t-v)^k e^{-sv}da(v).$$

The following theorem is proved: Let β_k be the abscissa of (C, k)-summability of the integral (*) and let $f(s)$ be the function defined by this integral. If $m_k(\beta_k, t)$ is non-decreasing for $t\geqq T_0\geqq 0$, and if $f^{(r)}(\beta_k t)=0$, $r=1, \cdots, k$ (the rth right derivative), then $f(s)$ has a singularity at $s=\beta_k$. The conditions imposed for m_k and $f^{(r)}$ are in a certain sense necessary.

This theorem represents the correction of a theorem previously given by the author [Mitt. Math. Sem. Giessen no. 47 (1954); MR 16, 693]. *M. Tomić* (Belgrade)

3734:

Rao, V. V. L. N. On certain kernel functions. Gaṇita 9 (1958), 33–41.

The author uses known integrals to give several examples to Rule 2 on p. 268 of Titchmarsh's *Fourier integrals* [Clarendon Press, Oxford, 1937].

A. Erdélyi (Pasadena, Calif.)

3735:

Narain, Roop. A Fourier kernel. Math. Z. 70 (1958/59), 297–299.

Let $K(s)$ denote the Mellin transform of the Meijer function

$$2^{\frac{1}{2}}G_{2p,2q}^{q,p}\left(\frac{x^2}{4}\left|\begin{matrix}a_1, \cdots, a_{2p}\\ b_1, \cdots, b_{2q}\end{matrix}\right.\right).$$

A Meijer function can be defined as an integral of a ratio of two Gamma function products, and from this definition the author proves the following result.

If $a_j+a_{p+j}=\frac{1}{2}$ $(j=1, \cdots, p)$ and $b_j+b_{q+j}=\frac{1}{2}$ $(j=1, \cdots, q)$, then $K(s)K(1-s)=1$. If these conditions are satisfied the Meijer function is then a Fourier kernel of a generalized Fourier transform.

C. Fox (Montreal, P.Q.)

3736:

Martirosyan, R. M. An integral transformation. Akad. Nauk Armyan. SSR. Dokl. 27 (1958), 65–74. (Russian. Armenian summary)

Let $\Psi(x, \lambda, \alpha)$ be the solution of $y''-2ia\lambda y'+\lambda^2 y=0$ such that $\Psi(0, \lambda, \alpha)=\sin\alpha$ and $\Psi'(0, \lambda, \alpha)=-\cos\alpha$. Let ω_1 and ω_2 denote the two real numbers $\sqrt{(a^2+1)}\pm a$, respectively. Let $\Omega(\lambda)=\rho'(\lambda)$ where

$$\rho(\lambda) = i\omega_1(\omega_1+\omega_2)\lambda^2/4\pi.$$

For any $f\in L_2(0, \infty)$ let $\Phi(\lambda)=\text{l.i.m.}\int_0^N f(t)\exp(i\omega_2\lambda t)dt$ as $N\to\infty$. The purpose of this note is to verify the relations

$$f(x) = \text{l.i.m.}\int_{-N}^N \Phi(\lambda)\Psi(x, \lambda, \alpha)\Omega(\lambda)d\lambda$$

for $\cot\alpha<0$, and

$$f(x) = \text{l.i.m.}\int_{-N}^N \Phi\Psi\Omega d\lambda$$
$$+\omega_1^2(\omega_1^2+1)\cot(\alpha)\Phi(i\omega_1\cot\alpha)\cdot\exp(-\cot\alpha\cdot x).$$

A. N. Milgram (Berkeley, Calif.)

3737:

Ribarič, M. On the inversion of integral transforms. Arch. Rational Mech. Anal. 3 (1959), 45–50.

Consider the transform

$$(*)\qquad g(y) = \int_a^b K(y, x)f(x)dx,$$

where y ranges in some region S of the complex plane and g is analytic on S. An inversion f of (*) can be formally obtained as follows: since

$$g^{(j)}(y_0) = \int_a^b K^{(j)}(y_0, x)f(x)dx,$$

the numbers $g^{(j)}(y_0)$ can be considered to be inner products of $f(x)$ and $\overline{K^{(j)}(y_0, x)}q(x)$ where q is a suitable weight function and inner product (φ, ψ) is defined as

$$\int_a^b \varphi(x)\overline{\psi(x)}[q(x)]^{-1}dx.$$

From the functions $\overline{K^{(j)}(y_0, x)}q(x)$ construct an orthonormal set φ_m. Then the Fourier coefficients (f, φ_m) of f are just linear combinations of the $g^{(j)}(y_0)$ and hence f can be obtained from the $g^{(j)}(y_0)$. Stringent conditions making the above process valid are given. A similar argument is applied to obtain f from $g(y_j)$ for some sequence y_j in S. Special cases involving the Laplace transform are due to Erdélyi and Tricomi. *R. R. Goldberg* (Evanston, Ill.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 3546.

3738:

★**Smithies, F. Integral equations.** Cambridge Tracts in Mathematics and Mathematical Physics, no. 49. Cambridge University Press, New York, 1958. x+172 pp. $4.50.

The preface of the book opens with the statement that "the present work is intended as a successor to Maxime Bôcher's tract, *An introduction to the study of integral equations* [University Press, Cambridge, 1909]." This remark is indeed an understatement, since the present work is much more than a continuation of Bôcher's treatment, good as it was fifty years ago. It must have taken a tremendous amount of reading and sifting of the enormous literature to give a modernized version of the theory in this little volume of scarcely more than 150 pages.

There may be those who would disagree with the author's decision to omit many of the well-known applications of the theory of integral equations. Thus we find no mention of Abel's problem, of singular equations, of boundary value problems, of Green's function or of the various applications to problems of theoretical physics. These the reader may find in books by Hamel and the more recent Russian books which have been translated into English. The reviewer feels that the author showed good judgment in refraining from giving what would have had to be a sketchy treatment of these various applications.

Throughout the book the author uses complex-valued functions of a real variable. The functions are measurable and the integrals are taken in the sense of Lebesgue. To take care of the square integrable functions and kernels the author has recast much of the classical theory of Fredholm. The reader who expects a superficial modification of the usual theory will be surprised to find substantial changes. One such change is that, wherever possible, the author obtains equality everywhere; whereas most previous writers who have used the Lebesgue integral in this context have been content with equality almost everywhere.

It should be noted that, while the theory is not presented in terms of linear operators, the author has made extensive use of the notations of operator theory in order to shorten many of the formulas. The reviewer was glad to see that the concept of relatively uniform convergences was used. This was first introduced by E. H. Moore over fifty years ago in his treatment of integral equations in an abstract space. By means of Moore's concept the author is able to generalize the usual treatment of Neumann series and the analogue of the Hilbert-Schmidt expansion theorem.

The theorems of Fredholm are first taken up by the use of "determinant free" methods due to Schmidt and Radon. The author begins with kernels of finite rank, or degenerate kernels, as they are sometimes called. He writes them in the form of "direct product" borrowed from matrix theory. If the reader should happen to omit the place in Chapter III where the notation is first introduced, he should refer to the index of notations given at the end of the book.

Using the method of polynomial approximation for two variables, the author writes the general kernel as a sum of a kernel of finite rank and a kernel of small norm. The first leads to a finite system of ordinary linear equations and the second may be dealt with by the use of the Neumann series. Of course, in this treatment, the explicit formulas of the solution of the integral equation cannot be given in terms of the kernel itself.

Next, the usual results of orthogonalization of the square integrable functions of one and two variables are given, with a treatment of the Riesz-Fischer theorem. This is in preparation for the classical theory and the application to the Hermitian kernels. First the author considers the theory for continuous kernels, and then for square integrable kernels for which $K(t, t)$ is measurable and summable. This permits him to develop the Fredholm formula for L^2 kernels without the assumption of the existence of a trace in the usual sense. Here, as in the rest of the tract, the author is again to be commended for a very careful and excellent account of the theory which he himself has modified.

In the chapter on Hermitian kernels as well as in the following one, the method is that used by Erhard Schmidt [Math. Ann. **63** (1907), 433–476; **64** (1907), 161–174], which the author extended in 1937 [Proc. London Math. Soc. (2) **43** (1937), 255–279]. There is an application to positive continuous kernels (Mercer's theorem). The chapter closes with an account of the non-homogeneous equation with Hermitian kernels but without the use of the general Fredholm theory, analogous to the method used by Schmidt for symmetric kernels.

Having shown that an Hermitian L^2 kernel is determined up to equivalence by its systems of characteristic values and characteristic functions, the author now takes up the extension to the general L^2 kernel. If $K(s, t)$ is a non-null L^2 kernel, and the L^2 functions u and v and the complex number μ satisfy the relations $u=\mu Kv$, $v=\mu K^*u$, then μ is said to be a singular value of K, and the pair u, v are called singular functions of K belonging to the singular value μ. Here, $Kv=\int_a^b K(s, t)v(t)dt$ and K^* is the Hermitian adjoint of K. The author extends Schmidt's theory of

non-symmetric kernels to these general kernels. Some interesting applications of these results are given to the approximation of L^2 kernels of finite rank, to the theory of normal kernels ($KK^* = K^*K$), and to the theory of linear integral equations of the first kind.

The reviewer believes that this book gives a masterful treatment of the subject. It should be suitable as an introduction for a beginner, provided he knows some of the more elementary properties of functions which are Lebesgue integrable. These results he could easily acquire by looking up the references given by the author. It seems strange that in the excellent bibliography the recent books of Tricomi [*Integral equations*, Interscience Publ., New York-London, 1957; MR **20** #1177] and Mikhlin [*Integral equations*, Pergamon Press, New York-London-Paris-Los Angeles, 1957; MR **19**, 428] are not mentioned; but the reviewer was told by the author that these books came out too late to be included.

I. A. Barnett (Cincinnati, Ohio)

3739:

Gel'fond, A. O. On estimation of certain determinants and the application of these estimations to the distribution of eigenvalues. Mat. Sb. N.S. **39** (81) (1956), 3–22. (Russian)

Nach Ergebnissen von A. O. Gel'fond, Hille und Tamarkin ist die Fredholmsche Determinante $D(\lambda)$ zur Integralgleichung $f(x) = \lambda \int_0^1 K(x, y) f(x) dy$ von der Ordnung von $\exp(\theta \lg |\lambda|)$, wenn $K(x, y)$ analytisch in y auf der abgeschlossenen Strecke $\langle 0, 1 \rangle$ ist. Dies bleibt nicht mehr richtig, wie der Verf. zeigt, wenn auf die Analytizität in den Endpunkten verzichtet wird, da für $K(x, y) = (1-x)^m(1-y)^m/(1-xy)$ die Relation gilt: $D(-r) > \exp(\pi^{-1} m^{-1} \lg^3 r)$ $(r > r_0(m))$ (Satz 6). Unter gewissen speziellen Voraussetzungen liefert aber $\exp(\theta \lg^3 |\lambda|)$ eine Abschätzung für $|D(\lambda)|$, wie der Verf. in seinem Satz 9 beweist. Er setzt dabei im wesentlichen voraus, daß $K(x, y) = O((xy(1-x)(1-y))^{\delta-1/2})$ ist für ein δ, $0 < \delta \leq \frac{1}{2}$, und $0 < x < 1$, $0 < y < 1$, sowie daß $K(x, y)$ analytisch in y in einem gewissen möndchenförmigen Bereich ist. Der Beweis dieser Sätze beruht auf verschiedenen sehr subtilen Abschätzungen der verallgemeinerten Vandermondeschen Determinanten

$$\Delta_{n+1}\left(\frac{x_0, \cdots, x_n}{\alpha_0, \cdots, \alpha_n}\right) = \begin{vmatrix} x_0^{\alpha_0} & \cdots & x_0^{\alpha_n} \\ \cdot & \cdots & \cdot \\ x_n^{\alpha_0} & \cdots & \alpha_n^{\alpha_n} \end{vmatrix}$$

sowie der Integrale über solche. Zur Charakterisierung dieser Resultate sei der Satz 5 des Verf. angegeben: Für ein natürliches m sei allgemein $\alpha_k = e^{\pi\sqrt{k}} + \varphi(k)$, wo $\varphi(k) \geq 0$, $\varphi(k) = o(k^{-4} e^{\pi\sqrt{k}})$ und $\alpha_k - \alpha_{k-1} \geq m+1$ ist. Dann gilt für ein konstantes γ

$$\int_0^1 \cdots \int_0^1 \prod_{s=0}^{n} (1-x_s)^{2m} \Delta_{n+1}^2\left(\frac{x_0, \cdots, x_n}{\alpha_0, \cdots, \alpha_n}\right) dx_0 \cdots dx_n > \exp\left(-\tfrac{4}{3}(3m+2)\pi n^{3/2} - \gamma n \lg n\right).$$

A. Ostrowski (Zbl **70**, 329)

3740:

Mihaĭlov, L. G. An investigation of a new type of two-dimensional integral equations. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 27–30. (Russian)

The solution of a certain singular integral equation is reduced to the determination of an analytic function satisfying certain conditions.

H. P. Thielman (Ames, Iowa)

3741:

Aširov, S. Conditions of complete continuity of an operator *K*. Azerbaĭdžan. Gos. Univ. Uč. Zap. **1958**, no. 6, 13–25. (Russian. Azerbaijani summary)

Let $p = \{p_1, \cdots, p_n\}$ be a vector with $p_i \geq 1$ $(i = 1, \cdots, n)$. Denote by L_p the space of vector functions $\mathbf{u}(x) = \{u_1(x), \cdots, u_n(x)\}$ with $u_i(x) \in L^{p_i}(G)$, where G is some bounded measurable set of a finite dimensional Euclidean space. The norm for a vector in L_p is defined as the sum of the norms of its components. By generalizing a theorem of Krasnosel'skiĭ and Ladyženskiĭ [Trudy Moskov. Mat. Obšč. **3** (1954), 307–320; MR **15**, 966] the author obtains sufficient conditions for a non-linear integral operator of the form $K\mathbf{u} = \{K_1\mathbf{u}, \cdots, K_n\mathbf{u}\}$, where

$$K_i\mathbf{u} = \int_G K_i[x, s; u_1(s), \cdots, u_n(s)] ds,$$

to take L_p into some $L_{\bar{p}}$ and be completely continuous. These sufficient conditions on K are as follows: (1) The functions $K_i(x, s, u_1, \cdots, u_n)$ are measurable in (x, s) and continuous in $(u_1, \cdots, u_n)$ for almost all (x, s); (2) for arbitrary bounded measurable functions $f_1(x, s), \cdots, f_n(x, s)$ the function

$$K_i[x, s; f_1(x, s), \cdots, f_n(x, s)]$$

belongs to $L^{p_i}(G \times G)$; (3) to each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that whenever meas $E < \delta$ and $E \subset G$ then

$$\int_G \left[\int_E |K_i(x, s; u_1(s), \cdots, u_n(s))| ds\right]^{\bar{p}_i} dx < \varepsilon,$$

for all $\mathbf{u} = (u_1(s), \cdots, u_n(s))$ in the unit sphere of L_p.

D. H. Hyers (Los Angeles, Calif.)

3742:

Nadžafov, K. A. Branch points of solutions of a class of non-linear integro-differential equations. Trudy Azerbaĭdžan. Indust. Inst. Azizbekov. **19** (1957), 202–225. (Russian. Azerbaijani summary)

A detailed study of the solutions of a very general non-linear integro-differential equation in a neighborhood of a given solution is made. The equation is:

$$(*) \quad F[\lambda, \varphi(x), \varphi'(x), \varphi''(x)] = \int_0^1 K(x, s, \lambda) f[\lambda, s, \varphi(s), \varphi'(s)] ds,$$

where F is analytic in its variables; K is analytic in λ; for each fixed λ, the function K is regular and symmetric in x and s where $s, x \in [0, 1]$; and f is continuous in s for $s \in [0, 1]$ and analytic in its other variables. It is assumed that λ_0 is an eigenvalue of multiplicity one of an appropriate linear operator and that (*) has solution $\varphi_0(x)$ if $\lambda = \lambda_0$. Then, "generally speaking", for each λ close enough to λ_0, equation (*) has two solutions each of which converges to $\varphi_0(x)$ when $\lambda \to \lambda_0$. The exact hypotheses are too long to be described here. The proof, which involves lengthy computations, consists in assuming a series form for the solution, solving for the coefficients in the series and proving that the resulting series converges.

J. Cronin (Elizabeth, N.J.)

FUNCTIONAL ANALYSIS

See also 3451, 3508, 3509, 3510, 3616, 3645, 3652, 3669, 3721, 3722, 3723, 3724, 3725, 3728, 3824.

3743:

Nakano, Hidegorô. On an extension theorem. Proc. Japan Acad. **35** (1959), 127.

The extension theorem of the author's earlier paper [same Proc. **33** (1957), 603–604; MR **20** #2957] is stated to be false, T. Itô having found a counter-example. A corrected statement is announced without proof.

3744:

Singer, Ivan. Remarque sur un théorème d'approximation de H. Yamabe. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. **26** (1959), 33–34.

The theorem of the title may easily be reformulated as follows: If M is a dense convex subset of a normed space E, and if G is a closed subspace of finite deficiency n, then $M \cap G$ is dense in G. An exercise in Bourbaki [*Espaces vectoriels topologiques*, Ch. 1, Actual. Sci. Ind. no. 1189, Hermann, Paris, 1953; MR **14**, 880; p. 55] states this result for $n=1$ and E a topological vector space. The author makes an n-fold application of Bourbaki's result, obtaining a simple proof of Yamabe's theorem for topological vector spaces. *R. R. Phelps* (Princeton, N.J.)

3745:

Inaba, Mituo. A note on coordinated spaces. Kumamoto J. Sci. Ser. A **3**, 189–194 (1958).

L'A. généralise d'abord un résultat de Banach aux espaces de Fréchet (par la même méthode de démonstration, basée sur le théorème du graphe fermé): si un espace de Fréchet admet une "base" (e_n), c'est-à-dire que tout $x=\sum_n f_n(x)e_n$, la série étant convergente, alors les f_n sont continues. Il étudie ensuite en substance les systèmes biorthogonaux dans un espace de Fréchet E et son dual E'; ses résultats sont des cas particuliers de théorèmes du référent [Michigan Math. J. **2** (1954), 7–20; MR **16**, 47] qui s'appliquent plus généralement aux espaces tonnelés E; par exemple son th. 4 est un cas particulier de la prop. 10, p. 13 du travail cité. *J. Dieudonné* (Paris)

3746:

★Kolmogorov, A. Dimension linéaire des espaces vectoriels topologiques. Séminaire Bourbaki; 10e année: 1957/1958. Textes des conférences; Exposés 152 à 168; 2e éd. corrigée, Exposé 165, 1 p. Secrétariat mathématique, Paris, 1958. 189 pp. (mimeographed)

The author urges the study of functions $d(E)$, defined for E any topological vector space, with values in a partially ordered set; subject to the conditions that $d(E_1) \leq d(E)$ in case E_1 is isomorphic to a closed subspace of E, or in case $TE=E_1$ for some continuous linear T. He offers some suggestions concerning the direction such a study should take. *C. Davis* (Providence, R.I.)

3747:

Raĭkov, D. A. A criterion of completeness of locally convex spaces. Uspehi Mat. Nauk **14** (1959), no. 1 (85), 223–229. (Russian)

With the aid of a theorem of Vilenkin and Graev the author proves the following (theorem 1): Let X be a locally convex Hausdorff linear space, covered by a sequence (F_n) of absolutely convex subsets for which always $F_n+F_n \subset F_{n+1}$. Suppose the absolutely convex neighborhoods of 0 in X are exactly those absolutely convex sets V for which $V \cap F_n$ is always a neighborhood of 0 in F_n (under the relative topology). Then X is complete if, for each n, every Cauchy filter in F_n is convergent in X.

Now suppose (X, i) is the inductive limit of a sequence $(X_n, \tau_n)_{n=1,2,\cdots}$, where as usual X is the union of the increasing sequence (X_n) of its linear subspaces, each (X_n, τ_n) is a locally convex Hausdorff linear space, the injection map $(X_n, \tau_n)\to(X_{n+1}, \tau_{n+1})$ is always continuous, and i is the finest locally convex topology for X under which the injection $(X_n, \tau_n)\to(X, i)$ is always continuous. With $F_n=X_n$, the above theorem yields at once the well-known criterion of Köthe [Math. Z. **52** (1950), 627–630; MR **12**, 417] for completeness of the space (X, i). Also deduced is a theorem of Grothendieck on completeness of (DF)-spaces [Summa Brasil. Math. **3** (1954), 57–123; MR **17**, 765]. With the aid of theorem 1 it is proved that (X, i) is complete if each of the injections $(X_n, \tau_n)\to(X_{n+1}, \tau_{n+1})$ is weakly completely continuous (and thus in particular if each space (X_n, τ_n) is a reflexive Banach space). For complete continuity this was proved by Sebastião e Silva [Rend. Mat. e Appl. (5) **14** (1955), 388–410; MR **16**, 1122]. *V. L. Klee, Jr.* (Copenhagen)

3748:

Colojoară, Ion. Sur une topologie définie à l'aide des familles de quasi-semi-normes. Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 371–383. (Romanian. Russian and French summaries)

Let E be a vector space over K, K being the real or complex field. A quasi-seminorm of order r $(0<r<+\infty)$ on E is a real-valued function q on E which is subadditive and such that

$$q(\alpha x) = |\alpha|^r q(x) \quad (x \in E,\ \alpha \in K).$$

A subset A of E is said to be quasi-convex of order s $(0<s<+\infty)$ if $t^sA+(1-t)^sA \subset A$ whenever $0 \leq t \leq 1$. A topology on E is said to be quasi-locally convex of order s if it is compatible with the vector space structure of E, and if there exists a base at 0 formed of sets which are symmetric and quasi-convex of order s.

The relationship between quasi-seminorms of order r and quasi-locally convex topologies of order $s=1/r$ is exhibited (theorems 1 and 3): the results are as one might expect from knowledge of the usual case $s=r=1$. In addition (theorem 2), the upper bound of quasi-locally convex topologies of fixed order s is a topology of the same type. A quasi-seminorm q is said to be a quasi-norm if $q(x)>0$ whenever $x \neq 0$. Then (theorem 4) any quasi-locally convex space of order s is the projective limit of quasi-normed spaces of order s. *R. E. Edwards* (Reading)

3749:

Altman, M. Continuous transformations of a locally convex space onto itself. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 37–39.

The writer replaces the metric sphere of radius ε by a symmetric neighborhood of the origin and so extends the

Borsuk ε mapping notion to locally convex linear topological spaces. He obtains generalizations of some theorems of Borsuk and Granas and of an earlier result of his own.

D. G. Bourgin (Urbana, Ill.)

3750:

Subba Rao, M. V. A property of transformations over a sequence of spaces. J. Indian Math. Soc. (N.S.) **22** (1958), 59–64.

Let (τ_α) be a family of vector space topologies on a vector space E such that, given any two, there is a third finer than each, and let τ denote the upper bound topology on E (the coarsest topology finer than each τ_α). Also let (Γ_β) be a family of locally bounded vector space topologies on another vector space V and Γ their upper bound topology. The author proves, first in a special case and then generally, that a linear mapping f of E into V is continuous under τ and Γ if and only if for each β there is an α for which f is continuous under τ_α and Γ_β.

A. P. Robertson (Lawrence, Kans.)

3751:

Subba Rao, M. V. On the existence of a norm weaker than a given family of norms on a vector space. J. Indian Math. Soc. (N.S.) **22** (1958), 53–58.

The author considers a sequence (N_i) of norms on a vector space E and shows that there is a norm topology on E finer than that defined by each N_i if and only if there are positive constants a_i with $\sum a_i N_i(x)$ convergent for each x in E. He deduces that, on the space of entire functions, or the space of functions analytic in a circle, there is no norm topology finer than the topology of compact convergence.

A. P. Robertson (Lawrence, Kans.)

3752:

Klee, Victor. On the Borelian and projective types of linear subspaces. Math. Scand. **6** (1958), 189–199.

Each separable Banach space is shown to contain, for each $\gamma<\omega$, a dense subspace which is projective exactly of class γ. For each $\beta<\Omega$, there exists a dense subspace of E which is borelian of additive class β but not of multiplicative class β. The proof is based on the following idea. If $\mathscr{F}$ is a family of metric spaces satisfying certain conditions (similar to those defining borel or projective classes) and if $S\in\mathscr{F}$ is a subset of a linear metric space, then the linear extension of S is shown to belong to $\mathscr{F}$ as well.

V. Pták (Prague)

3753:

Dinculeanu, Nicolae; et Foiaş, Ciprian. Mesures vectorielles et opérations linéaires sur L_E^p. C. R. Acad. Sci. Paris **248** (1959), 1759–1762.

Let m be a countably additive measure defined on the Borel sets of a locally compact space T with values in a Banach space X. It is assumed that the total variation μ of m is finite. (I) If E, G are Banach spaces and $X=\mathscr{L}(E,G)$, the authors define an integral with respect to m over functions in $L_E{}^1(\mu)$ (= the μ-integrable functions from T to E), first for simple functions and then by continuous extension. (II) If m is regular and if $X=\mathscr{L}(E,F')$, where E and F are separable, then there is a μ-unique mapping U_m of T into X such that $\|U_m(t)\|=1$ and

$$\langle z, \int x dm\rangle = \int \langle z, U_m(t)x(t)\rangle d\mu(t)$$

for z in F and x in $L_E{}^1(\mu)$. (III) A similar type of representation is given if m is absolutely continuous with respect to a positive measure ν. (IV) A representation of the same form is given for a linear transformation f from $L_E{}^p(\nu)$ to F' satisfying a boundedness condition of the form $\sup \sum \|f(\phi_{A_i}a_i)\|<\infty$ (where ϕ_A denotes the characteristic function of the Borel set A). These results extend earlier work of the authors. No proofs are given.

R. G. Bartle (Urbana, Ill.)

3754:

Davis, Harry F. On isosceles orthogonality. Math. Mag. **32** (1958/59), 129–131.

Let V be a real normed space; the elements v and w are said to be isosceles orthogonal if $\|v+w\|=\|v-w\|$, and the set $C(v)$ of all elements isosceles orthogonal to v is called the orthogonal complement of v. If V is Euclidean (i.e., if the norm in V is derived from an inner product), then isosceles orthogonality is equivalent to ordinary orthogonality, and in this case $C(v)$ is a linear subspace for every v. The following theorem is a particular case of a stronger theorem due to R. C. James [Duke Math. J. **12** (1945), 291–302; MR **6**, 273]: A real normed linear space V is Euclidean if and only if $C(v)$ is a linear subspace for every $v\in V$. The present paper contains an elementary proof for this particular case.

A. C. Zaanen (Leiden)

3755:

Cheney, Ward; and Goldstein, Allen A. Proximity maps for convex sets. Proc. Amer. Math. Soc. **10** (1959), 448–450.

Let K be a closed convex set in a Hilbert space H. The proximity map P for K is the map which assigns to each point $x\in H$ the unique point Px of K nearest to x. Let K_1, K_2 be two closed convex sets in H, let P_i be the proximity map for K_i $(i=1,2)$, and $Q=P_1P_2$. The following results are proved. (I) x is a fixed point of Q if and only if $x\in K_1$ and $\|x-P_2x\|$ is the distance between K_1, K_2. (II) Convergence of $\{Q^nx\}$ to a fixed point of Q is assured for every $x\in H$, when either (a) one of K_1, K_2 is compact, or (b) one of K_1, K_2 is finite-dimensional and the distance between them is attained. (III) In a finite-dimensional Euclidean space, if each of K_1, K_2 is the intersection of a finite number of closed half-spaces, then the distance between K_1, K_2 is attained.

Ky Fan (Notre Dame, Ind.)

3756:

Gould, G. G. On a class of integration spaces. J. London Math. Soc. **34** (1959), 161–172.

If E is an abstract measure space with positive measure μ, Λ is a linear space of measurable functions on E, and Λ' is the set of measurable functions $x'(t)$ $(t\in E)$ which for all $x\in\Lambda$ satisfy the condition $\int_E |x(t)x'(t)|d\mu<\infty$, then Λ is called an integration space with dual Λ'. For such spaces, a subset S of Λ is called (weakly) bounded if for each $x'\in\Lambda'$ the values of the bilinear form $\int x(t)x'(t)d\mu$ are bounded uniformly for all $x\in S$. Bounded sets in Λ' are similarly defined interchanging Λ and Λ'. Given a family $\mathfrak{S}$ of bounded subsets of Λ', the sets

$$U(S) = \left\{x\in\Lambda : \int |x(t)x'(t)|d\mu \le 1 \quad \text{for all} \quad x'\in S\right\}$$

define, for $S\in\mathfrak{S}$, a family of neighborhoods of zero in Λ

for a topology on Λ. If $\mathfrak{S}$ is the family of all finite [all bounded] subsets of Λ' this topology is called the weak [strong] topology. Interchanging Λ and Λ', weak and strong topologies on Λ' are similarly defined.

The principal object of the paper is a discussion of the integration space Ω, defined as the set of all measurable functions on E whose integral over any set of finite measure is finite, and its dual Ω'. In this discussion the measure μ is assumed to be non-atomic, and E assumed measurable with $\mu(E)=+\infty$. The space Ω is shown to be the direct sum of L^1 and L^∞ (considered as non-topological spaces), and Ω' to be the intersection of L^1 and L^∞. Bounded sets of Ω and Ω' are characterized in terms of properties of the functions they contain. The strong topology on Ω is shown to be that given by the norm $\|x\|=\sup \int_K |x(t)|d\mu$ where K varies over all sets of unit measure ($x\in\Omega$). The strong topology on Ω' is proved to be that defined by the norm $\|y\|=\max(M, N)$, $y\in\Omega'$, where $M=\text{ess sup } |y(t)|$ and $N=\int_E |y(t)|d\mu$. *W. R. Transue* (Gambier, Ohio)

3757:

Hausner, Alvin. A generalized Stone-Weierstrass theorem. Arch. Math. **10** (1959), 85–87.

Let $\mathfrak{U}$ denote an arbitrary finite-dimensional associative or non-associative algebra with basis $e_1, e_2, \cdots, e_n$ over the real numbers. $C(\Omega, \mathfrak{U})$ denotes the set of all continuous $\mathfrak{U}$-valued functions defined on the compact Hausdorff space Ω. If $\mathfrak{U}$ is the set of real numbers, then the Stone-Weierstrass theorem asserts that a complete subalgebra A of $C(\Omega, \mathfrak{U})$ is identical with $C(\Omega, \mathfrak{U})$ if A contains all constant functions and separates the points of Ω. If $\mathfrak{U}$ is the set of complex numbers it is known this theorem is no longer true in this form. Recently J. C. Holladay [Proc. Amer. Math. Soc. 8 (1957), 656–657; MR **19**, 293] proved that this theorem is true if $\mathfrak{U}$ is the real quaternion algebra. In this paper it is shown, by extending Holladay's method, that the Stone-Weierstrass theorem is true for all real Cayley-Dickson type algebras of dimension 2^n ($n>1$) and for all real Clifford algebras of dimension 2^n with n even. *W. A. J. Luxemburg* (Pasadena, Calif.)

3758:

Sauer, Robert. Introduzione nel calcolo delle distribuzioni con applicazioni ad un problema aerodinamico. Rend. Sem. Mat. Fis. Milano **28** (1959), 18–35. (English summary)

Expository paper of the main ideas of Schwartz' theory of distributions. The paper follows mainly the lines of H. Koenig and F. Penzlin. The many-dimensional case and the Marcel Riesz operator are considered together with an application to aerodynamics: the three-dimensional hypersonic current around a wing of finite length. *L. Cesari* (Lafayette, Ind.)

3759:

Guerreiro, J. Santos. Les changements de variable en théorie des distributions. I. Portugal. Math. **16** (1957), 57–81.

Let I be an open interval on the real line R; by $C_\pi(I)$ we denote the space of distributions on I. The author proves that any continuous linear map L of $C_\pi(I)$ into $C_\pi(I)$ is given by a change of variables, that is, there exists a C^∞ function h: $I\to I$ with $h'(x)\neq 0$ for $x\in I$ such that L is the (unique) extension to $C_\pi(I)$ of the map $f(x)\to f(h(x))$ for f continuous on I. A similar result is proven in case I is compact. *L. Ehrenpreis* (Waltham, Mass.)

3760:

McKibben, John J. Singular integrals in two dimensions. Amer. J. Math. **81** (1959), 23–36.

Let P be a polynomial in n real variables. Then L. Schwartz has posed the problem as to whether there exists a distribution T with $PT=1$. If this property holds for all P then we could even take T to be tempered [L. Schwartz, *Théorie des distributions*, tome I, Hermann, Paris, 1950; MR **12**, 31]. The author proves that this is actually the case for $n=2$. Moreover, he gives an explicit extension of the Hadamard "partie finie" procedure (too complicated to discuss here) to actually construct T. From the author's results it follows that every constant coefficient partial differential equation in 2 variables possesses a tempered fundamental solution. This improves (for $n=2$) results of Malgrange [Ann. Inst. Fourier Grenoble **6** (1956), 271–355; MR **19**, 280] and the reviewer [Amer. J. Math. **76** (1954), 883–903; MR **16**, 834].

{The existence of a T for each P has recently been proven (though non-constructively) for all n by S. Łojasiewicz and L. Hörmander.} *L. Ehrenpreis* (Waltham, Mass.)

3761:

Fenyő, Stefan. Über den Zusammenhang zwischen den Mikusińskischen Operatoren und den Distributionen. Math. Nachr. **19** (1958), 161–164.

Mikusiński [*Rachunek operatorów*, Polskie Towarzystwo Matematyczne, Warsaw, 1953; MR **16**, 243] calls operators the elements obtained by forming a field from the ring of functions on $(0, \infty)$ with convolution taken as multiplication. It is shown that every distribution of finite rank with support in $(0, \infty)$ is an operator; and that every operator of the form a/b is such a distribution provided that $b(t)=t^r\beta(t)$ with $r>0$, $\beta(+0)\neq 0$. *J. L. B. Cooper* (Cardiff)

3762:

Fenyő, István. On the connection between Mikusiński operators and distributions. Magyar Tud. Akad. Mat. Fiz. Tud. Oszt. Közl. 8 (1958), 385–392. (Hungarian)

A Hungarian version of the article reviewed above.

3763:

Dieudonné, Jean. Sur les espaces L^1. Arch. Math. **10** (1959), 151–152.

The space $L^1(\mu)$, where μ is the Lebesgue measure on $\langle 0, 1\rangle$, is shown to be not isomorphic to an adjoint space to a Banach space. Outline of the proof: If L^1 were an adjoint of F, then F would be a closed separable subspace of L^∞. The theorem of Dunford-Pettis applied to the identity transform $L^1\to L^1=F'$ yields the existence of a kernel $K(s, t)$ such that

$$\int g(t)\varphi(t)dt = \int g(t)\int K(s, t)\varphi(s)dsdt$$

for each $g\in L^1$ and each $\varphi\in F$. Since L^1 is separable, the unit sphere of L^∞ is metrisable in $\sigma(L^\infty, L^1)$. It follows

that each $f \in L^\infty$, $|f| \leq 1$, is a limit of a sequence $\varphi_n \in F$, $|\varphi_n| \leq 1$.

By a simple limit process, the above equality may be extended for each $\varphi \in L^\infty$. It follows that

$$f(t) = \int K(s, t) f(s) ds$$

almost everywhere for each $f \in L^\infty$. This is impossible as may be immediately seen on taking for f_n the characteristic function of $\langle 0, x_n \rangle$, where x_n is a sequence dense in $\langle 0, 1 \rangle$.
V. Pták (Prague)

3764:

Lamperti, John. On the isometries of certain function-spaces. Pacific J. Math. **8** (1958), 459–466.

Characterizations of the linear, norm-preserving operators on L_p and l_p, $1 \leq p\ (\neq 2) < \infty$, appear in Banach's book. New proofs of more general results are given in the present paper. The underlying set is taken as any σ-finite measure space (X, F, μ), and the norm is replaced by the functional

$$I(f) = \int \Phi(|f(x)|) d\mu(x),$$

where Φ is a continuous strictly increasing function on $[0, \infty)$ with $\Phi(0) = 0$ and $\Phi(\sqrt{t})$ either a convex or a concave function of t. Crucial to the argument is the following lemma, based on results of Karamata [Bull. Soc. Math. France **61** (1933), 55–62]: with Φ as above and z, w any complex numbers, the inequality

$$\Phi(|z+w|) + \Phi(|z-w|) \geq 2\Phi(|z|) + 2\Phi(|w|)$$

holds for $\Phi(\sqrt{t})$ convex in t, and the reverse inequality holds for $\Phi(\sqrt{t})$ concave in t; if the convexity or concavity is strict, equality holds when and only when $zw = 0$. It follows that for f, g in L_p, $0 < p\ (\neq 2) < \infty$, the equality $\|f+g\|_p{}^p + \|f-g\|_p{}^p = 2\|f\|_p{}^p + 2\|g\|_p{}^p$ holds if and only if $fg = 0$ a.e. This enables the author to answer an open question of Boas, by showing that there does not exist an isometric linear mapping of H_p onto L_p provided $0 < p < \infty$ and $p \neq 2$.

A regular set isomorphism of (X, F, μ) is a mapping T of F into itself defined modulo sets of measure zero and satisfying (i) $T(X-A) = TX - TA$, (ii) $T(\bigcup_{n=1}^\infty A_n) = \bigcup_{n=1}^\infty TA_n$ for disjoint A_n, (iii) $\mu(TA) = 0$ if and only if $\mu(A) = 0$. A regular set isomorphism induces a linear transformation (denoted also by T) on the set of measurable functions, the characterizing property being $T\varphi_A = \varphi_{TA}$, where φ_A is the characteristic function of the set A. The author proves a number of general theorems connecting regular set isomorphisms and I-isometries; for simplicity we state the main result in the setting of L_p. Theorem: Let U be a linear operator on L_p, $0 < p\ (\neq 2) < \infty$, such that $\|Uf\|_p = \|f\|_p$ for all $f \in L_p$. Then there exists a regular set isomorphism T and a function h in L_p such that $Uf = h \cdot Tf$ and $|h|^p = d\mu^*/d\mu$ a.e. on TX, where $\mu^*(A) = \mu(T^{-1}A)$. Conversely, if T is a regular set isomorphism and h is defined by $|h|^p = d\mu^*/d\mu$, then $\|Uf\|_p = \|f\|_p$ for all $f \in L_p$.
M. G. Arsove (Seattle, Wash.)

3765a:

U, Cun-sin′. Perfect spaces and perfect matrix rings. I. Boundedness and convergence of a perfect matrix ring. Sci. Record (N.S.) **3** (1959), 95–102. (Russian)

3765b:

U, Cun-sin′. Perfect spaces and perfect matrix rings. II. A theorem on the product in a perfect matrix ring. Sci. Record (N.S.) **3** (1959), 103–106. (Russian)

Let λ be a perfect Köthe-Toeplitz space of sequences of complex numbers, $\Sigma(\lambda)$ the set of all matrices which map λ into itself [Köthe and Toeplitz, J. Reine Angew. Math. **171** (1934), 193–226]. The author discusses different definitions of boundedness, and of convergence of sequences of matrices from $\Sigma(\lambda)$. He shows the equivalence of some of the possible definitions; in the second paper this equivalence is connected with the correctness of a "product theorem" for matrices of $\Sigma(\lambda)$.
G. G. Lorentz (Syracuse, N.Y.)

3766:

Amemiya, Ichiro; and Shiga, Kôji. On tensor products of Banach spaces. Kōdai Math. Sem. Rep. **9** (1957), 161–178.

A. Grothendieck [vgl. Bol. Soc. Mat. São Paulo **8** (1956), 1–79; MR **20** #1194] entwickelte eine Theorie der Tensorprodukte von Banachräumen, ohne jedoch die Beweise im einzelnen auszuführen. Verff. geben für einige der Hauptresultate vollständige Beweise und gelangen an einer Stelle zu einer Verschärfung eines Grothendieckschen Resultats.
G. Köthe (Zbl **79**, 324)

3767:

Goldberg, Seymour. Linear operators and their conjugates. Pacific J. Math. **9** (1959), 69–79.

The various relations between the requirement that an operator have a bounded inverse, inverse, dense range, etc., and the corresponding requirements for its adjoint, are completely enumerated: first for general linear operators with dense domain, then for closed operators. Examples are given of the cases which are possible.
J. T. Schwartz (Berkeley, Calif.)

3768:

Pták, Vlastimil. On approximate solutions of linear equations in Banach spaces. Časopis Pěst. Mat. **83** (1958), 389–398. (Russian. Czech and English summaries)

The idea is to approach $(E-H)^{-1}$, E being the identity and H a bounded linear operator on a Banach space, via such operators as $(E-PH)^{-1}$, $(E-HP)^{-1}$, $(E-PHP)^{-1}$, the main case being that in which P is a finite-dimensional projector. Results are proved connecting the existence and the values of these inverses. Also considered is the "distance of an operator from a subspace"; the application of this concept to approximate solutions is to be dealt with in future papers of the author.
F. V. Atkinson (Canberra)

3769:

Lidskiĭ, V. B. Non-selfadjoint operators with a trace. Dokl. Akad. Nauk SSSR **125** (1959), 485–487. (Russian)

A linear operator C in a separable Hilbert space H is said to have a trace when $\sum_{i=1}^\infty (C\varphi_i, \varphi_i) < \infty$ for one—and hence every—orthonormal basis φ_i. The set of all such operators forms a two-sided ideal in the algebra of all linear bounded operators in H. Theorem 1: $\sum_{i=1}^\infty (C\varphi_i, \varphi_i) = \sum_{s=1}^\infty \lambda_s$, where λ_s is the sequence of all eigenvalues of C. Theorem 3: If T is a bounded quasi-nilpotent operator, completely continuous, and such that T^k has a trace, then

for any two non-zero elements f and g of H the entire function $((E-\lambda T)^{-1}f, g)$ is of order k and minimal type. A result is also given on the completeness of eigenfunctions and generalized eigenfunctions in the sense of Livšic and Mukminov. *G.-C. Rota* (Cambridge, Mass.)

3770:

Müller, P. Heinz. Eine neue Methode zur Behandlung nichtlinearer Eigenwertaufgaben. Math. Z. **70** (1958/59), 381–406.

The author studies the equation

$$(*)\qquad f-\lambda Af-\lambda^2 Bf+\lambda^2\sum_{k=1}^{n}(\lambda-a_k)^{-1}H_k f = 0,$$

and an inhomogeneous variant; here $f\in R$, a Hilbert space, and the operators A, B, H_k are symmetric and completely continuous, $B\geqq 0$, $a_k^{-1}H_k\geqq 0$. He establishes the reality and discreteness of the spectrum, and also expansion theorems. The special case in which the H_k are of finite rank had been considered by D. F. Harazov [Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **21** (1955), 145–168; MR **17**, 1216], using different methods. The main idea of the present paper is to re-write (*) as a linear eigen-value problem $(\mathfrak{E}-\lambda\mathfrak{R})\mathfrak{f}=0$ in the product space R^{n+2}; here $\mathfrak{R}$ is an $(n+2)$-by-$(n+2)$ matrix of operators on R. Complications arise from the fact that the author's choice of substitutions leads to an unsymmetric matrix $\mathfrak{R}$. It is however shown that $\mathfrak{R}$ can be symmetrised, considering it as an operator on a certain quotient space of R^{n+2}, with metric depending on the B, H_k. There is a very full bibliography. *F. V. Atkinson* (Canberra)

3771:

Lin, Chün. On the calculation of the eigenvalues for functional equations. Sci. Record (N.S.) **3** (1959), 180–184.

The author considers the approximation of the eigenvalues of a symmetric operator H on a Hilbert space X by those of an operator $\bar{H}$ on a (usually finite dimensional) Hilbert space $\bar{X}$, which is isomorphic to a subspace X' of X.

H is approximated by an operator T on X with range in X'. By the isomorphism φ this operator induces an operator $\varphi T\varphi^{-1}$ on $\bar{X}$ which, in turn, approximates $\bar{H}$. By a theorem of Weyl, the eigenvalues of $\bar{H}$ lie within $\|H-T\|+\|\varphi T\varphi^{-1}-\bar{H}\|$ of those of H.

The operator T, the space X', and the isomorphism φ serve only as intermediaries and are not uniquely determined by H and $\bar{H}$. Thus the author obtains error estimates for the method of mechanical quadratures for integral operators by introducing either trigonometric polynomials or piecewise constant functions.

The results are closely related to those of H. Wielandt [Proc. Sympos. in Appl. Math. vol. VI, pp. 261–282, McGraw-Hill, New York, 1956; MR **19**, 179].

H. F. Weinberger (College Park, Md.)

3772:

Markus, A. S. On holomorphic operator-functions. Dokl. Akad. Nauk SSSR **119** (1958), 1099–1102. (Russian)

Let A_λ be a closed linear operator on one Banach space $\mathfrak{B}_1$ into another, $\mathfrak{B}_2$, A_λ being also a holomorphic function of λ in some open region G of the complex plane. For such λ let A_λ be also a Φ_+-operator, i.e. have a closed range and admit only a finite number $\alpha(A_\lambda)$ of linearly independent solutions of $A_\lambda x=0$, $x\in\mathfrak{B}_1$. Then $\alpha(A_\lambda)$ is constant in G except for an isolated set, at which it may take greater values. The dual result is also given. The proof uses solutions $x(\lambda)$ of $A_\lambda x=0$ which are holomorphic in λ. This generalises work of the reviewer [Acta Sci. Math. Szeged **15** (1953), 38–56; MR **15**, 134], for more restricted values of A_λ, and of S. N. Kračkovskiĭ [Dokl. Akad. Nauk SSSR **96** (1954), 1101–1104; MR **16**, 263], for A_λ depending linearly on λ. *F. V. Atkinson* (Canberra)

3773:

Foiaş, Ciprian. On Hille's spectral theory and operational calculus for semi-groups of operators in Hilbert space. Compositio Math. **14**, 71–73 (1959).

Let A be a linear operator with domain D in Hilbert space H for which there is a real number r such that every real number $\xi>r$ is in $\rho(A)$ and $|(A-\xi I)^{-1}|\leqq(\xi-r)^{-1}$. Let $\bar{O}_r$ be the algebra of all complex continuous functions φ defined on $\operatorname{Re}\lambda\leqq r$ and holomorphic on $\operatorname{Re}\lambda<r$. Using his previous results on spectral measures the author establishes an operational calculus which maps $\bar{O}_r$ into the algebra $L(H)$ of bounded linear operators in H. The calculus has the properties that (a) the map $\varphi\to\varphi(A)$ is an isomorphism; (b) $|\varphi(A)|\leqq\sup_{\operatorname{Re}\lambda=r}|\varphi(\lambda)|$; (c) if φ_n is bounded in $\bar{O}_r$ and $\varphi_r(\lambda)\to\varphi(\lambda)$ uniformly on every compact contained in $\operatorname{Re}\lambda\leqq r$ then $\varphi_n(A)\to\varphi(A)$ strongly; (d) e^{tA} is a strongly continuous semi-group and A is its generator. Furthermore these properties uniquely determine the map of $\varphi\to\varphi(A)$. *N. Dunford* (Brooklyn, N.Y.)

3774:

Greub, Werner; and Rheinboldt, Werner. On a generalization of an inequality of L. V. Kantorovich. Proc. Amer. Math. Soc. **10** (1959), 407–415.

If A is a linear selfadjoint operator in Hilbert space H, E the identity operator, and $0<mE\leqq A\leqq ME$ (inequalities signify that the differences are semidefinite), then for any $x\in H$,

$$(x,x)^2\leqq(Ax,x)(A^{-1}x,x)\leqq(x,x)^2(M+m)^2/(4Mm).$$

If also B is linear selfadjoint and is permutable with A, and $0<m'E\leqq B\leqq M'E$, then, for any $x\in H$,

$$(Ax,Ax)(Bx,Bx)\leqq(Ax,Bx)^2(MM'+mm')^2/(4mm'MM').$$

The first inequality generalizes one of Kantorovich for finite matrices obtained from the spectral decomposition; the second, in the finite case, when expressed in terms of the spectral decomposition, was stated by Pólya and Szegö. The above forms are proved and shown to be equivalent. The Kantorovich inequality is important in numerical analysis for establishing the rate of convergence of the method of steepest descent.

A. S. Householder (Oak Ridge, Tenn.)

3775:

Solomyak, M. Z. Application of semigroup theory to the study of differential equations in Banach spaces. Dokl. Akad. Nauk SSSR **122** (1958), 766–769. (Russian)

A study is made of the abstract Cauchy problem in Banach spaces:

$$(1)\qquad \frac{du}{dt}+Au=f(t),\quad u(0)=u_0,$$

where $u(t)$ and $f(t)$ are elements of a complex Banach space and A is a closed linear operator with domain dense in the Banach space. Under the conditions that the resolvent set of A contains some sector $\Sigma: \varphi \leqq \arg \lambda \leqq 2\pi - \varphi$ ($\varphi < \pi/2$) and $\|R(\lambda; A)\| \leqq c/(|\lambda|+1)$ for λ in Σ, A generates a semigroup $\{T(t)\}$. The sense in which $u(t) = T(t)u_0 + \int_0^t T(t-\tau)f(\tau)d\tau$ is a solution of (1) is studied. This generalizes results of Krasnosel'skiĭ, Kreĭn, and Sobolevskiĭ in Hilbert spaces [same Dokl. **112** (1957), 990–993; MR **19**, 747]. With weaker conditions on A and stronger conditions on u_0 and $f(t)$, this problem has been studied by R. S. Phillips [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 244–248; Trans. Amer. Math. Soc. **74** (1953), 199–221; MR **15**, 880; **14**, 882].
J. P. LaSalle (Baltimore, Md.)

3776:

Bergström, Harald. Konvergenzsätze über unendliche Produkte in abstrakten Halbgruppen (Verallgemeinerungen der klassischen Grenzwertssätze in der Wahrscheinlichkeitsrechnung). Abh. Math. Sem. Univ. Hamburg **23** (1959), 228–256.

The author investigates necessary and sufficient conditions for the convergence of infinite products in certain types of semi-groups. The conditions are mainly concerned with the convergence of corresponding infinite sums. Questions of divisibility (in the probability sense) are also considered. Although the conditions are reasonably complicated, they are natural for the applications to the theory of distribution functions which the author gives.
J. Blackman (Syracuse, N.Y.)

3777:

Hausner, Alvin. On generalized group algebras. Proc. Amer. Math. Soc. **10** (1959), 1–10.

Let G be a locally compact abelian group and X a complex commutative Banach algebra with identity e. Denote by $B(G, X)(=B)$ the group algebra consisting of X-valued Bochner integrable functions on G with respect to the Haar measure of G, in which the multiplication is defined by convolution as in the L_1-group algebra $L(G)$, and the norm in B is defined by the L_1-integral of the pointwise X-norm. The correspondence theorem between bounded *-representation of $L(G)$ onto Hilbert space and continuous unitary representation of G is generalized for B, in Theorem 1, and the Stone-Ambrose-Godement theorem on unitary representations of locally compact abelian groups [e.g., Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, Toronto-New York-London, 1953; MR **14**, 883] is proved for a product representation (continuous unitary for G and bounded for X) in Theorem 2. In the final part, the author discusses the relations of kernels and hulls between ideals of B and X, where X has not necessarily e, proving the Segal-Kaplansky-Helson theorem in a Loomis' formulation (cf. the book mentioned above) for certain restricted ideals in B. When G is compact, an imbedding theorem (of X into B) is proved. And when X is an abelian group algebra, some remarks and examples are given. {Reviewer's remark: Theorem 1 may be proved without the assumption of commutativity of G and X by slightly modifying the proof of the author and Theorem 2 is an easy consequence of the known spectral representation of X on its maximal ideal space and the Stone-Ambrose-Godement theorem.}
H. Umegaki (Tokyo)

3778:

Civin, Paul. A maximum modulus property of maximal subalgebras. Proc. Amer. Math. Soc. **10** (1959), 51–54.

Let B be a complex commutative regular Banach algebra with identity e and space of maximal ideals $\mathfrak{M}$. Let π: $x \to x(M)$ be the Gelfand representation of B. A subalgebra A is called non-determining if $\pi(A)$ is not dense in $\pi(B)$. For this notion see Civin and Yood [same Proc. **7** (1957), 1005–1010; MR **18**, 586]. Let A be a maximal non-determining subalgebra of B which is not a maximal ideal of B and separates points of $\mathfrak{M}$. It is first observed that $e \in A$, so that A has a compact space $\mathfrak{N}$ of maximal ideals. Each $M \in \mathfrak{M}$ determines in a natural way a point of $\mathfrak{N}$. It is shown that this mapping is a homeomorphism of $\mathfrak{M}$ into $\mathfrak{N}$ whose image is the Šilov boundary of $\mathfrak{N}$.
B. Yood (Eugene, Ore.)

3779:

Żelazko, W. On a certain class of topological division algebras. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 201–203. (Russian summary, unbound insert)

A preliminary discussion, and the statement of a central theorem, concerning sufficient conditions upon a complete metric division algebra in order that it be isomorphic with either the real field, the complex field, or the division algebra of real quaternions. Earlier criteria, due severally to Mazur, Arens and the reviewer, are obtained as special cases. The proofs and further developments are to appear in Studia Mathematica.
R. E. Edwards (Reading)

3780:

Arens, Richard. Dense inverse limit rings. Michigan Math. J. **5** (1958), 169–182.

The author, continuing the work of his paper in the Pacific J. Math. **2** (1952), 455–471 [MR **14**, 482], studies the linear algebras called $\mathscr{F}$-algebras by Michael [Mem. Amer. Math. Soc. no. **11** (1952); MR **14**, 482]. Inverse mapping systems are used to prove the main theorem which, in a slightly weakened form (the original is for a class of topological rings) is as follows. "Let A be an $\mathscr{F}$-algebra with a unit element 1 and let J be a proper right ideal with finitely many generators, in A. Then there is a continuous homomorphism T of A into a Banach algebra B with unit such that $T(J)$ lies in some proper right ideal of B."

The joint right spectrum $\sigma(a_1, a_2, \cdots, a_N; A)$ of a finite set $(a_n)_{1\leqq n\leqq N}$ of elements of A is the set of complex N-vectors $(\lambda_n)_{1\leqq n\leqq N}$ such that the right ideal J in A generated by the set $(a_n - \lambda_n)_{1\leqq n\leqq N}$ is $\neq A$. It is characterised in terms of the possible B and T and, in the commutative case, is shown to be identical with the image of Δ (=the family of continuous homomorphisms of A into the complex numbers) under the map $\zeta \to (\zeta(a_n))_{1\leqq n\leqq N}$.

If Ω is a non-empty open plane set, Hol Ω denotes the $\mathscr{F}$-algebra of holomorphic functions on Ω. Let A be an $\mathscr{F}$-algebra with identity, and suppose that A is generated by $z \in A$ in the sense that the smallest closed inverse-closed subalgebra of A containing z is A. Suppose that A has a continuous derivation D such that $Dz = 1$, and that positive constants $r_0, r_1, \cdots$ exist such that

$$\|D^k f\|_n < k! r_n^{-k} \|f\|_{n+1} \quad (n = 0, 1, 2, \cdots; k = 1, 2, \cdots),$$

where $\|\cdot\|_n$ is the nth semi-norm used in defining A. Then

it is shown that A is commutative and, if N is the radical, that A/N is homeomorphically isomorphic with Hol Ω for some Ω, and that this provides a characterization of the algebras Hol Ω. Generalizing a result of Helmer [Duke Math. J. **6** (1940), 345–356; MR **1**, 307], the author also proves that finitely generated ideals in Hol Ω are principal.
D. A. Edwards (Oxford)

3781:

Nakamura, Masahiro. A proof of a theorem of Takesaki. Kōdai Math. Sem. Rep. **10** (1958), 189–190.

From the fact that an ascending weak* limit of normal positive linear functionals on a von Neumann algebra A is again normal, it follows by a straightforward application of Zorn's lemma that any positive linear functional on A is the sum of a normal one and one which majorizes no normal one.
J. Feldman (Berkeley, Calif.)

3782:

Shimoda, Isae. Notes on general analysis. VII. J. Gakugei Tokushima Univ. Math. **9** (1958), 19–20.

[For part VI, see same J. **7** (1956), 1–8; MR **19**, 434.]

Suppose E_1 and E_2 are complex Banach spaces. Suppose f is an analytic mapping from $U=\{x:\|x\|<1\}$ (in E_1) into E_2. Suppose f maps U in a one-to-one manner onto a domain D_f in E_2, the inverse mapping f^{-1} being analytic on D_f. Suppose $f(0)=0$, $\sup_{x\in U}\|f(x)\|\leq M<\infty$. Finally, suppose the expansion of $f^{-1}(y)$ near 0 is $\sum_{n=1}^{\infty} g_n(y)$, g_n being a homogeneous polynomial of degree n, and g_1 in particular being a linear operator. Let $K\geq\|g_1\|$. The main assertion of the paper seems to be that D_f contains a sphere $V=\{y:\|y\|<\rho\}$, where ρ is a positive number depending only on M and K, ρ being otherwise independent of f. What is actually proved is that there exist positive numbers ε, δ, depending only on M and K, such that $\delta<1$ and $\|f(x)\|\geq\varepsilon\|x\|$ if $\|x\|\leq\delta$. This implies the assertion about V, with $\rho=\varepsilon\delta$, if X is finite-dimensional, but the reviewer does not feel sure of the implication otherwise.
A. E. Taylor (Los Angeles, Calif.)

3783:

Granas, A. On continuous mappings of open sets in Banach spaces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 25–29. (Russian. English summary)

Theorem: Let f be a mapping from an open subset O of a Banach space X into X having the form $f(x)=x-F(x)$, where F is completely continuous. Suppose that for each $x\in O$, there exists $\varepsilon_x>0$ such that the ε_x-neighborhood of x is contained in O, and for any x', x'' in that neighborhood, $f(x')=f(x'')$ implies $\|x'-x''\|<\varepsilon_x$. Then $f(O)$ is open in X. Various theorems on invariance of domain are derived from this theorem. The proof of the theorem depends upon Krasnosel'skiĭ's [Ukrain. Mat. Ž. **3** (1951), 174–183; MR **14**, 1109] generalization to Banach spaces of Borsuk's theorem on antipodes [cf. M. Altman, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 293–295; MR **20** #3531].
M. Jerison (Lafayette, Ind.)

3784:

Altman, Mieczyslaw. Une nouvelle preuve du théorème de l'invariance du domaine dans les espaces à voisinages convexes. C. R. Acad. Sci. Paris **246** (1958), 2094–2095.

The theorem: in a locally convex l.t.s., a homeomorphism which differs from the identity by a completely continuous function takes open sets onto open sets. It is due to Leray [J. Math. Pures Appl. (9) **24** (1946), 201–248; MR **7**, 468]. Granas has obtained this and other results in the case of Banach spaces as corollaries from a systematic treatment of mappings [see the above review]. The present note extends enough of Granas' theory to locally convex spaces to yield the cited theorem.
J. Isbell (Seattle, Wash.)

3785:

Mibu, Yoshimichi. Cauchy's theorem in Banach spaces. J. Math. Soc. Japan **11** (1959), 76–82.

The author proves the Cauchy theorem in the following setting: X, Y, Z are three complex Banach spaces; there is a "multiplicative-like" distributive and homogeneous mapping from $X\times Y$ to Z, $(x, y)\to x\circ y=z$, satisfying also the condition $\|x\circ y\|\leq\|x\|\cdot\|y\|$. The function f is defined on an open convex subset D of X, has values in Y and possesses a Fréchet differential $\delta f(x; h)$, $x\in D$, $h\in X$. It satisfies the condition—critical for the proof—$\delta f(x; h)\circ k=\delta f(x; k)\circ h$ for all $h, k\in X$, $x\in D$. Then $\int_\Gamma f(x)\circ dx=0$ for any rectifiable curve Γ in D. This result contains as a special case a theorem of the reviewer for functions analytic over a Banach algebra [Trans. Amer. Math. Soc. **54** (1943), 414–425; MR **5**, 100] where the definition of analyticity satisfies the condition above, as well as an older result for analytic functions of a complex variable with values in a Banach space.
E. R. Lorch (New York, N.Y.)

3786:

Kwan, Chao-chih; et Lin, Chün. Sur la méthode d'équations approximatives pour résoudre des équations fonctionnelles non-linéaires. Sci. Record (N.S.) **1** (1957), 385–389.

Die Verfasser übertragen eine Idee von L. V. Kantorovitch [Uspehi Mat. Nauk (N.S.) **3** (1948), no. 6 (28), 89–185; MR **10**, 380] zur Behandlung linearer Gleichungen auf nichtlineare Gleichungen. Eine in einem Banachraum X gegebene Gleichung (1) $p(x)=x-f(x)=0$ wird durch eine Gleichung (2) $\bar{p}(\bar{x})=\bar{x}-\bar{f}(\bar{x})=0$ in einem Banachraum $\bar{X}$ approximiert. Die Verfasser nennen Bedingungen, unter denen das vereinfachte Newtonsche Verfahren für die Näherungsgleichung (2) konvergiert, und geben eine Abschätzung für Lösungen der Gleichung (1) an, welche die Näherungen jenes Verfahrens enthält. Bei einem Beispiel mit (1) als Integralgleichung und (2) als System gewöhnlicher Gleichungen werden Bedingungen genannt, unter denen die allgemeinen Voraussetzungen erfüllt sind.
J. Schröder (Hamburg)

GEOMETRY

See also 3345, 3813, 3814, 3873.

3787:

Goormaghtigh, R. Sur les angles formés par une droite et les côtés d'un triangle. Mathesis **67** (1958), 338–343.

θ_a, θ_b, θ_c are the angles made by a line with the oriented sides a, b, c of the triangle ABC, the orientation induced

by the order A, B, C. O is the circumcenter, R the circumradius of the circumcircle Γ of triangle ABC.

By making use of the Argand plane the following known relation is derived.

$$\sum \sin 2A \cos 2\theta_a = 0.$$

By similar methods five or six additional relations are obtained some of which are presumed new. E.g., if T is any point on the circumcircle; H the orthocenter of the triangle formed by the symmetrics of T with respect to OA, OB, OC; and H_2' the projection of H on OT; then

$$\sum \cos 2A \sin 2\theta_a = \frac{HH_2'}{R},$$

$$\sum \cos 2A \cos 2\theta_a = -\frac{OH_2'}{R}.$$

L. M. Kelly (East Lansing, Mich.)

3788:

Thébault, Victor. Sphères associées à un tétraèdre. Mathesis **68** (1959), 46–51.

Il s'agit des vingt-quatre sphères passant par trois sommets et tangentes à une arête d'un tétraèdre $ABCD$ dont les arêtes ont pour mesures $BC=a$, $DA=a'$, $CA=b$, $DB=b'$, $AB=c$, $DC=c'$. Nous considérerons successivement six quaternes de ces sphères associées aux quadrangles

$$Q_1 \equiv ABCD, \quad Q_2 \equiv ABCD, \quad Q_3 \equiv ABCD,$$

les sphères en cause passant par trois sommets et tangentes à l'un des côtés restants et obtenues en parcourant le périmètre de ces quadrangles dans deux sens opposés.

Résumé de l'auteur

3789:

Thébault, Victor. Some formulas pertaining to a tetrahedron. Amer. Math. Monthly **66** (1959), 407–408.

Si dans un tétraèdre a et a' désignent les longueurs de deux arêtes opposées, cot a, cot a' les cotangentes des angles dièdres correspondants, l'expression

$$a^2+a'^2+2aa' \cot a \cot a'$$

a la même valeur H pour les trois couples d'arêtes opposées. L'auteur donne une expression de H en fonction des longueurs des arêtes, du rayon R de la sphère circonscrite et du volume V; il en résulte que H peut s'exprimer en fonction uniquement des longueurs des arêtes.

M. Decuyper (Lille)

3790:

Goormaghtigh, R. Sur les droites de Simson d'un polygone inscriptible. Mathesis **67** (1958), 241–245.

P_1 is a polygon of n vertices inscribed in a circle Γ, P_2 the pedal polygon of any point M of Γ with respect to P_1, P_3 the pedal polygon of M with respect to P_2 and so on. The $(n-2)$nd polygon P_{n-2} degenerates (vertices are linear). In this note it is proved that if all but two of the vertices remain fixed and the remaining two vary in such a manner that the distance between them is constant, then the envelope of the line of P_{n-2} is a circle, the center of which is midway between M and the Simson line of M with respect to the polygon defined by the $n-2$ fixed points. Various corollaries of this theorem are noted.

L. M. Kelly (East Lansing, Mich.)

3791:

Viola, Tullio. Poligoni equivalenti per traslazione. Period. Mat. (4) **36** (1958), 310–321.

Two sets P and P' of polygons are equivalent by translation if there exist finite non-overlapping decompositions $P=\bigcup_1^n P_\nu$ and $P'=\bigcup_1^n P_\nu'$ such that each polygon P_ν is congruent by translation to P_ν'. Hadwiger and Glur gave a necessary and sufficient condition for two polygons to be equivalent by translation [Elem. Math. **6** (1951), 97–106; MR **13**, 576; in particular "Kriterium T", p. 106]. In the present paper their result is reformulated, extended to two sets of polygons, and given a new proof.

P. Scherk (Boulder, Colo.)

3792:

★Kramer, Werner. Darstellende Geometrie. I. Hochschulbücher für Mathematik, Bd. 38. VEB Deutscher Verlag der Wissenschaften, Berlin, 1959. viii+188 pp. DM 12.50.

This book is a text on descriptive geometry intended for students of technical universities and technical schools; it may also be useful in secondary school teaching. The present first part is a conventional introduction to orthogonal projection on one or more planes; a second part will deal with parallel projection in general, central projection and axonometry. Table of contents:—I. Senkrechte Parallelprojektion auf eine Bildebene. 1. Punkt, Gerade und Ebene. 2. Dächer, Böschungen und Sonnenuhren. 3. Kreis, Kreiszylinder und Kreiskegel. 4. Die Kugel. 5. Anwendungen auf die mathematische Erd- und Himmelskunde. 6. Normalrisse als Hilfsebenen.—II. Senkrechte Parallelprojektion auf mehrere Bildebenen. 7. Punkt, Gerade, Ebene, ebenflächige Gebilde. 8. Durchdringung von Polyedern. 9. Kreis, Kegel, Zylinder, Kugel.—III. Ebene Schnitte des geraden Kreiskegels und ihre senkrechten Projektionen. 10. Die Ellipse. 11. Die Parabel. 12. Die Hyperbel. 13. Abwicklung des Mantels von Rotationszylindern und Rotationskegeln.—There are 221 clear figures and numerous exercises.

F. A. Behrend (Melbourne)

3793:

Bakos, T. Octahedra inscribed in a cube. Math. Gaz. **43** (1959), 17–20.

The author solves a problem proposed by Dorman Luke [same Gaz. **41** (1957), 189–194; MR **20** #3495; p. 194]: to inscribe in a cube a regular octahedron whose 6 vertices are one on each of 6 edges of the cube. Since each face of the cube contains an edge of the octahedron, the figure is self-dual. For a given cube there are 4 inscribed octahedra. Their 32 faces bound a convex solid which is derived from a cuboctahedron by placing a low square pyramid on each square face. Dually, for a given octahedron there are 4 circumscribed cubes. Their 32 vertices belong to a convex solid which is derived from a rhombic dodecahedron (the reciprocal of the cuboctahedron) by truncating the 6 vertices at which 4 faces meet. [For an equilateral variety of the same solid, see Coxeter, *Regular polytopes*, Methuen, London, 1948; MR **10**, 261; p. 32, Plate II, Fig. 5; or E. S. Fedorov, *Načala učeniya o figurah*, Izdat. Akad. Nauk SSSR, Moscow, 1953; MR **15**, 923; p. 276, Fig. 115.] This completes the solution of Luke's problem; but the author goes on to consider what he incorrectly calls the "dual problem": to inscribe in an octahedron a cube whose 8 vertices are one on each of 8 edges of the

octahedron. Since each face of the octahedron contains an edge of the cube, this figure is likewise self-dual. There are 3 cubes for each octahedron, 3 octahedra for each cube.

H. S. M. Coxeter (Toronto, Ont.)

3794:

Marchaud, André. Convexité et connexité linéaire. C. R. Acad. Sci. Paris **248** (1959), 2843–2844.

Two theorems concerning projective convexity are announced without proof. Theorem 1: The common frontier of two linearly connected complementary sets in the projective plane, each containing three non-linear points, is either an oval or a pair of lines. Theorem 2: The common frontier of two linearly connected complementary sets in projective three space, each of which contains four non planar points, is either (a) an ovoid, (b) the surface of a convex cone, (c) a pair of planes, or (d) a proper ruled quadric (i.e., not a quadric cone).

Definitions. A set is linearly connected if with each two of its points it contains one of the two complementary segments defined by these points. An oval is a Jordan curve in a plane meeting each line either in a segment or at most two points. An ovoid is a simple Jordan surface intersecting each line either in a segment or at most two points. A proper convex set in the plane is a non-linear closed linearly connected set which fails to intersect at least one line of the plane. A degenerate convex set in the plane is a closed set bounded by two lines. A proper convex set in projective 3 space is a linearly connected closed set, non-planar, which fails to intersect some plane. A degenerate convex set in 3 space is one consisting of points on those lines joining a given point to those of a plane convex domain, the given point not lying in this plane. If the plane convex domain is proper the set is a cone.

The author observes that in cases (a) and (b) of theorem 2 the closures of only one of the sets is convex, in case (c) the closures of both are and in (d) neither closure is.

L. M. Kelly (East Lansing, Mich.)

3795:

André, Johannes. Affine Ebenen mit genügend vielen Translationen. Math. Nachr. **19** (1958), 203–210.

An affine plane whose translations are transitive on the points is called a translation plane. A T-plane is a plane which has for each direction m a non-trivial group T_m of translations. A T^*-plane is a T-plane in which all the groups T_m are isomorphic. Gleason [Amer. J. Math. **78** (1957), 797–807; MR **18**, 593] has shown that a finite T^*-plane is a translation plane. In this paper it is shown that an infinite T^*-plane need not be a translation plane. An example is given in which the points are given by coordinates (x, y), x, y rational. Lines are of the Moulton type, i.e., viewed in the Euclidean plane are made up of broken line segments. Every group T_m is isomorphic to the additive group of integers. Thus the plane is a T^*-plane, but not a translation plane.

Marshall Hall, Jr. (Pasadena, Calif.)

3796:

★Tallini, Giuseppe. Una proprietà grafica caratteristica della superficie di Veronese negli spazi finiti. Convegno internazionale: Reticoli e geometrie proiettive, Palermo, 25–29 ottobre 1957; Messina, 30 ottobre 1957, pp. 136–139. Edito dalla Unione Matematica Italiana con il contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1958. vii+141 pp. 1800 Lire.

Denote by $S_{r,q}$ the finite projective geometry of dimension r coordinatized by GF(q). The surface of Veronese is a surface in $S_{5,q}$ with homogeneous coordinates $X_{ij}=X_{ji}$, $i, j=0, 1, 2$, given by the parametric equations $X_{ij}=x_ix_j$, $x_i, x_j \in \mathrm{GF}(q)$, $i, j=0, 1, 2$. There is an obvious one-one correspondence between the points of the surface of Veronese and the points (x_0, x_1, x_2) of $S_{2,q}$.

The present note gives a summary of the proof of the following. If a collection of k ($\geq q^2+q+1$) planes of $S_{r,q}$ (q odd) are incident two by two in such a way that no more than two pass through any one point, then $r=5$, $k=q^2+q+1$ and the planes comprise the set of tangent planes to a surface of Veronese. The proof in extenso is to be found in Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 19–23, 135–138 [MR **21** #1970].

J. D. Swift (Los Angeles, Calif.)

3797:

★Lunelli, L.; e Sce, M. Sulla ricerca dei k-archi completi mediante una calcolatrice elettronica. Convegno internazionale: Reticoli e geometrie proiettive, Palermo, 25–29 ottobre 1957; Messina, 30 ottobre 1957, pp. 81–86. Edito dalla Unione Matematica Italiana con il contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1958. vii+141 pp. 1800 Lire.

We recall that a k_q is a set of k points, of a finite projective plane over a GF(q), no three of which are collinear. A k_q is said to be complete when there is no $(k+1)_q$ containing it.

The present paper reports how the existence of complete 10_{11}, 12_{13}, 14_{17} has been proved, by the use of an electronic calculating machine.

B. Segre (Rome)

3798:

Pickert, Günter. Bemerkungen über die projektive Gruppe einer Moufang-Ebene. Illinois J. Math. **3** (1959), 169–173.

The author is concerned with the following questions: (1) if π is a Moufang plane, under what circumstances does the little projective group of π coincide with the projective group or the full collineation group of π, and (2) when does the little projective group share the property with the full collineation group of being transitive on (non-degenerate) quadrilaterals? A Moufang plane is a projective plane coordinatized by an alternative division ring; the plane is Desarguesian exactly when this alternative division ring is associative. In this latter case (Satz 1) the projective and little projective groups coincide exactly when the multiplicative group of the coordinatizing division ring is generated by its commutator group and the cubes of its center elements. If the alternative division ring is not associative, then it is a Cayley-Dickson algebra over its center (=ground field). Freudenthal has shown [*Oktaven, Ausnahmegruppen und Oktavengeometrie*, Matematisch Institut der Rijksuniversiteit te Utrecht, 1951; MR **13**, 433] that when this ground field is the field of real numbers, the two groups coincide and even equal the full collineation group (the exceptional group E_6). The author extends this as follows (Satz 2): if in the ground field every element is a cube and if the ground field can be so ordered that every positive element is a square, then the little projective group is transitive on quadrilaterals.

It is also shown (Satz 3) that the projective group of any Moufang plane is the product of the little projective group and a group of homologies with common center and axis.
D. R. Hughes (Chicago, Ill.)

3799:

Salzmann, Helmut. Viereckstransitivität der kleinen projektiven Gruppe einer Moufang-Ebene. Illinois J. Math. **3** (1959), 174–181.

This paper continues the investigations of the paper reviewed above. The author proves a number of theorems, such as: (1) if every element of the coordinatizing alternative division ring $\mathfrak{A}$ of a Moufang plane π has the form $((ab)c)((ab^{-1}a)c^{-1})$, then the little projective group of π is transitive on quadrilaterals; (2) if every element of $\mathfrak{A}$ is a cube, or if $\mathfrak{A}$ is generated by its commutators $bcb^{-1}c^{-1}$ and the cubes of its center elements, then the hypothesis of (1) is satisfied; (3) if $\mathfrak{A}$ is non-associative then the little projective group of π is transitive on quadrilaterals if for every quadratic extension $\mathfrak{K}$ of the ground field of $\mathfrak{A}$, every element of $\mathfrak{K}$ is a cube; (4) in the Desarguesian case, the little projective group is transitive on quadrilaterals exactly when $\mathfrak{A}$ is generated by the commutator group of its multiplicative group and the set of cubes in $\mathfrak{A}$. But he shows that there exist Desarguesian planes whose little projective group is transitive on quadrilaterals but is not equal to the (full) projective group (compare Satz 1 of the previous paper). Finally (5) he shows that the little projective group of a Moufang plane is transitive on quadrilaterals if the little projective group of every maximal Desarguesian subplane has this property.
D. R. Hughes (Chicago, Ill.)

3800:

Panella, Gianfranco. Isomorfismo tra piani di traslazione di Marshall Hall. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **25** (1958), 172–173.

Let F be a field and x^2-bx-c a quadratic polynomial irreducible over F. A Hall system $J(F,b,c)$ is the system of ordered pairs (x_1,x_2), $x_1, x_2\in F$ with the following addition and multiplication

$$(x_1,x_2)+(y_1,y_2) = (x_1+y_1, x_2+y_2),$$
$$(x_1,x_2)(y_1,0) = (x_1y_1, x_2y_1),$$
$$(x_1,x_2)(y_1,y_2) = \{y_1x_1-(y_1^2-by_1-c)y_2^{-1}x_2,\ y_2x_1 + (b-y_1)x_2\},$$

where $y_2\neq 0$. $J(F,b,c)$ is a Veblen-Wedderburn system which may be considered a quadratic extension of F in which every element $z\notin F$ satisfies $z^2=bz+c$. The author asks when two Hall systems $J(F,b,c)$ and $J(F,b',c')$ coordinatize the same projective plane. He relates this question to collineations in the three-dimensional (Desarguesian) space over F fixing a quadric. As a partial answer he shows that the two systems $J(F,b,c)$ and $J(F,b',c')$ give the same plane if F is the real field or a finite field of characteristic different from 2.
Marshall Hall, Jr. (Pasadena, Calif.)

3801:

★**Kustaanheimo, Paul. On the relation of order in finite geometries.** Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18–23 août 1957, pp. 139–140. Mercators Tryckeri, Helsinki, 1958. 209 pp. (1 plate)

An abstract of the article appearing in Rend. Mat. e Appl. (5) **16** (1957), 292–296 [MR **20** #1268].
Marshall Hall, Jr. (Pasadena, Calif.)

3802:

Smogorževs'kiĭ, O. S. Sur les angles du plan hyperbolique inscrits dans un cercle. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 463–464. (Ukrainian. Russian and French summaries)

Soit k une circonférence du plan hyperbolique. Sur l'arc ASB de cette circonférence, où $\bigcup AS=\bigcup SB$ et $\frac{1}{2}\pi<\angle AOS<\pi$, O étant le centre de k, existent deux tels points X et Y que $\angle AXB=\angle AYB=\frac{1}{2}\angle AOB$.

Soit C un point d'arc ASB, distinct de X et Y; alors $\angle ACB<\frac{1}{2}\angle AOB$, si C est un point d'arc XSY, et $\angle ACB>\frac{1}{2}\angle AOB$ dans le cas contraire.
Résumé de l'auteur

3803:

Andrièvs'ka, M. G. A common perpendicular of two real intersecting lines of a Lobachevsky space. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 465–467. (Ukrainian. Russian and English summaries)

The author proves that two real intersecting lines of a Lobachevsky space have two common perpendiculars—a real and an ideal.

The proof is based on Beltrami's interpretation.
Author's summary

3804:

Denisko, S. V. Equiareal interpretation of a Lobachevsky plane. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 574–577. (Ukrainian. Russian and English summaries)

This paper deals with two depictions of a Lobachevsky plane on a Euclidian plane, such that the areas of the depicted figures are retained. These depictions allow a ready solution of certain problems of Lobachevsky's geometry.
Author's summary

3805:

Fulton, Curtis M. A pseudo-Euclidean geometry. Proc. Amer. Math. Soc. **10** (1959), 304–306.

The author develops a "pseudo-Euclidean" geometry, based on the usual axioms for (real) Euclidean spaces, except that the null element is not considered a point and that the "inner product" satisfies $(x,y)=-(y,x)$. It is proved that the "dimension number" of this geometry is 2; the beginnings of trigonometry and analytic geometry are presented, the unrestricted duality of points and lines is pointed out and a model for the geometry (using elements of the Euclidean plane) indicated.
B. Grünbaum (Princeton, N.J.)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 3755, 3900, 3903a–c.

3806:

Bilinski, Stanko. Über die Ordnungszahl der Klassen Eulerscher Polyeder. Arch. Math. **10** (1959), 180–186.

In the notation of Hilbert and Cohn-Vossen [*Geometry and the imagination*, Chelsea, New York, 1952; MR **13**,

766; p. 309], the connectivity h of a surface is 1 for the sphere, 2 for the elliptic plane, 3 for the torus, and so on. According to Heawood's formula, any map on a surface of connectivity h can be colored with N_h colors, where $N_h=[(7+\sqrt{[24h-23]})/2]$.

Let V, E and F denote the numbers of vertices, edges and faces of an Eulerian polyhedron. Two such polyhedra are said to belong to the same class if they agree in all three properties. The author arranges the classes in a conventional order: one precedes another if its E is smaller, or if its E is the same while its V is smaller. Thus the first four classes are typified by the tetrahedron, square pyramid, triangular dipyramid, and triangular prism. He proves that a polyhedron in the hth class has N_h+2 edges.

H. S. M. Coxeter (Toronto, Ont.)

3807:

Maserick, Peter H. The structure of translation half rings in Euclidean spaces. Proc. Amer. Math. Soc. **10** (1959), 133–139.

A collection $\mathscr{R}$ of subsets of E^n is called a translation half ring (t.h.r.) provided it satisfies the following conditions: (i) if $R\in\mathscr{R}$ and $p\in E^n$, then $R+p\in\mathscr{R}$; (ii) if $R\in\mathscr{R}$ and $R'\in\mathscr{R}$, then $R\cap R'\in\mathscr{R}$; (iii) if R, $R'\in\mathscr{R}$ and $R\subset R'$, then there exist $R_1, R_2, \cdots, R_k\in\mathscr{R}$ such that $R=R_1\subset R_2\subset\cdots\subset R_k=R'$, and $R_j-R_{j-1}\in\mathscr{R}$ for $j=2, 3, \cdots, k$. A set $P\subset E^n$ is called a convex polyhedron provided it is bounded and is the intersection of a finite family of halfspaces, each of which is either open or closed. It is proved that if $\mathscr{S}$ is a t.h.r. of bounded convex subsets of E^n, then the closure of each member of $\mathscr{S}$ is a convex polyhedron. On the other hand, the set of all convex polyhedra in E^n is a t.h.r., as is the set of all parallelepipeds and degenerate polyhedra having extreme supports in n specified directions. Some other interesting results are also obtained.

V. L. Klee, Jr. (Copenhagen)

3808:

Míšek, Bohuslav. On the simplex polygon with the greatest volume of its convex hull. Časopis Pěst. Mat. **84** (1959), 99–104. (Czech. Russian and English summaries)

The author considers the "$(n+1)$-polygon A" in the euclidean space E_n, i.e., the closed polygon $A_1A_2\cdots A_{k+1}A_1$ in E_n which for preassigned circumference has maximum volume V_n of its convex hull. By elementary arguments, leaning on Schoute's *Mehrdimensionale Geometry* [vol. 1, Göschen, Leipzig, 1902], he establishes the fact that all sides have equal length a and all angles between any two sides A_jA_{j+1}, A_kA_{k+1} are equal to α such that $\cos\alpha=n^{-1}$. Also angles between sides and diagonals are studied and finally he finds that

$$V_n=(a^n/n!)[(n+1)^{n-1}/n^n]^{1/2}.$$

H. Schwerdtfeger (Montreal, P.Q.)

3809:

Barthel, Woldemar. Zum Busemannschen und Brunn-Minkowskischen Satz. Math. Z. **70** (1958/59), 407–429.

Généralisation intéressante d'un théorème de Busemann [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 27–31; MR **10**, 395], au moyen d'une inégalité, invariante dans l'espace affine, obtenue par une opération de composition linéaire, dépendant d'un paramètre, entre deux ensembles, dont l'un est convexe, l'autre compact, dont les mesures de dimension n sont positives, et qui sont situés dans deux demi-espaces R_n d'un espace R_{n+1}, ayant en commun un R_{n-1} et n'étant pas dans le prolongement l'un de l'autre; les deux ensembles donnés ayant la même intersection avec le R_{n-1} commun aux deux R_n, la mesure de dimension $(n-1)$ de cette intersection étant positive.

Examen de l'égalité et du cas particulier du théorème de Brunn-Minkowski.

Un énoncé équivalent est obtenu par une opération de symétrisation (Spiegeltheorem) et conduit à l'examen d'une loi de composition, appelée combinaison linéaire harmonique de deux ensembles, dont les propriétés sont étudiées.

J. Favard (Paris)

3810:

Turán, Pál. Certain types of extreme value problems. I, II. Köz. Mat. Lapok **16** (1958), 65–69, 97–101. (Hungarian)

The two articles contain two lectures delivered to an audience of high-school pupils. The problem of determining the best position in which to place a single ash-tray for the use of three smokers sitting at a table is formulated mathematically as a minimum-maximum problem, and solved. Other minimum-maximum problems are discussed briefly.

G. A. Dirac (Hamburg)

3811:

Fejes Tóth, L. An extremal distribution of great circles on a sphere. Publ. Math. Debrecen **6** (1959), 79–82.

This note contains one theorem and several conjectures concerning extremal properties of plane and spherical tessellations. [See Coxeter, *Regular polytopes*, Methuen, London, Pitman, New York, 1949; MR **10**, 261; for terminology.]

Theorem: If l denotes the length of the greatest edge of a spherical tessellation determined by $n>2$ great circles, then $l\geqq\pi/(n-1)$. Equality holds only for the three quasi-regular tessellations.

The inequality is obvious and in the case of the equality it follows readily that the tessellation is of type $(3, k, 3, k)$ from which the desired characterization is immediate. (Replace B' by C' in the second line of the equality argument.)

The author asks about the extremal configuration for large n and the lim inf nl. Also, he asks, are these questions connected somehow with the Euclidean tessellation $(3, 6, 3, 6)$. Other conjectures are noted.

L. M. Kelly (East Lansing, Mich.)

3812:

Heppes, A. Mehrfache gitterförmige Kreislagerungen in der Ebene. Acta Math. Acad. Sci. Hungar. **10** (1959), 141–148. (Russian summary, unbound insert)

A k-fold packing of circles in the plane is a set of congruent (unit) circles in the plane such that no point lies in the interior of more than k of the circles. A lattice packing is one in which the centers of the circles form a lattice. The least upper bound of the densities of k-fold packings is denoted by d_k' and that of lattice packings by d_k. The author proves by elementary geometric arguments that $d_k=kd_1=k\pi/\sqrt{12}$ for $k=1, 2, 3, 4$ and $d_k>kd_1$ for $k\geqq 5$. Since the author had previously shown [Elem. Math.

10 (1955), 125–127; MR 17, 526] that $d_k' > kd$ for $k > 1$, it follows that $d_k' > d_k$ for $k = 2, 3, 4$.
E. G. Straus (Los Angeles, Calif.)

3813:

Blumenthal, Leonard M. A budget of curiosa metrica. Amer. Math. Monthly **66** (1959), 453–460.

This is an expository article describing some elementary phenomena in Euclidean space not yet thoroughly understood. For example, while the three sides of a triangle "determine" that triangle (to within a congruence), the six edges of a tetrahedron do not so determine the tetrahedron. Indeed the six edges in some cases can be reassembled so as to produce $6!/4! = 30$ non-congruent tetrahedra. A set of six positive numbers is said to form a completely tetrahedral sextuple provided they are the lengths of 30 pairwise non-congruent tetrahedra. The numbers $(a+nd)^{1/2}$, $n = 0, 1, \cdots, 5$, $0 < 4d \leq a$, are shown to form a completely tetrahedral sextuple. For necessary and sufficient conditions see the next review.

A proof of an inequality conjectured by Pauc is presented. If 1, 2, 3 and 4 denote the vertices of a tetrahedron and e_i the sum of the two smallest angles of that face (triangle) opposite vertex i $(i = 1, 2, 3, 4)$, then $e_i + e_j + e_k \geq e_m$, where (i, j, k, m) is any permutation of $(1, 2, 3, 4)$.

A section is devoted to an interesting class of arcs in Hilbert space possessing neither a right- or left-hand tangent at any point, and a final section includes the simple but pretty theorem to the effect that at each point of a rectifiable metric arc the ratio of the arc to chord is unity if and only if that limit exists for each point.

There are some additional results too numerous to mention here, all elementary but none the less of interest and elegance. *L. M. Kelly* (East Lansing, Mich.)

3814:

Herzog, Fritz. Completely tetrahedral sextuples. Amer. Math. Monthly **66** (1959), 460–464.

This note provides solutions to the problem suggested by the work of L. M. Blumenthal of characterizing completely tetrahedral sextuples (c.t.s.). [See the preceding review.] A simple sufficient condition reads as follows: If $a > b > c > d > e > f > 0$ are the six numbers with $e + f \geq a$, then they are a c.t.s. provided $f \geq 2^{-1/2}a$, and (in a certain sense) $2^{-1/2}$ is the "best" constant possible.

Theorem: $a > b > c > d > e > f > 0$ are a c.t.s. if and only if $e + f \geq a$ and the metric quadruple with faces (triangles) (a, c, d) (a, e, f) (b, c, e) and (b, d, f) is congruently embeddable in E_3.

A second simple necessary and sufficient condition is also established. *L. M. Kelly* (East Lansing, Mich.)

GENERAL TOPOLOGY, POINT SET THEORY

See also 3354, 3355, 3505, 3506, 3535, 3783, 3784, 3839, 3840, 3847, 3848.

3815:

Koutský, Karel; and Sekanina, Milan. On the decomposition of the straight line in the congruent three-point sets. Časopis Pěst. Mat. **83** (1958), 317–326. (Czech. Russian and English summaries)

It is shown that to every three-point subset M of the real line there corresponds a decomposition R of the line such that $X \in R$ implies X is congruent to M (in the sense of sets). *V. E. Beneš* (Murray Hill, N.J.)

3816:

Shimrat, M. Simply disconnectible sets. Proc. London Math. Soc. (3) **9** (1959), 177–188.

The aim of this paper is to characterize both topologically and metrically those connected, locally connected, separable, metric spaces T which contain only separating points. This is accomplished by showing that every space T is acyclic and has a certain "tree-like" structure (the union of countably many rays which intersect in the expected manner). Whenever a space with this "tree-like" structure can be given a "length-metric" which agrees with its topology, the space must be a space T. On the other hand, a space is constructed (in the plane) which has this "tree-like" structure (and hence is arc-wise connected), which contains neither an end point nor a simple closed curve, and which is locally connected but not locally arc-wise connected. Obviously this space does not have a "length-metric" which agrees with its topology. Finally, the author characterizes a space T (above) as a locally arc-wise connected, connected metric space which is acyclic and contains no end point.

{In view of the vast literature (Moore, Wilder, Whyburn, Ayres, Gehman, and others) on problems of this kind dating back to 1920, the reviewer feels that the role of "completeness" would be of considerable interest and importance in the situation considered by the author.}
F. B. Jones (Chapel Hill, N.C.)

3817:

Rosen, Ronald H. Fixed points for multi-valued functions on snake-like continua. Proc. Amer. Math. Soc. **10** (1959), 167–173.

The multi-valued transformation $f: X \to Y$ is defined to be upper semi-continuous (u.s.c.) if, for each point $x \in X$, $f(x)$ is closed and, for each closed set $C \subset Y$, $f^{-1}(C)$ is closed; it is said to be lower semi-continuous (l.s.c.) if, for each open set $U \subset Y$, $f^{-1}(U)$ is open; finally, f is called continuous if it is both u.s.c. and l.s.c. The author proves several theorems about coincidences and fixed points for multi-valued transformations on chainable compact Hausdorff spaces. An example of his results is the theorem: if X and Y are chainable compact Hausdorff spaces, $f, g: X \to Y$ are u.s.c., f is onto, and $f(x)$ is connected for each $x \in X$, then there exist points $x \in X$ and $y \in Y$ such that $y \in f(x) \cap g(x)$. *E. Dyer* (Chicago, Ill.)

3818:

Bing, R. H. Each homogeneous nondegenerate chainable continuum is a pseudo-arc. Proc. Amer. Math. Soc. **10** (1959), 345–346.

Except for the possibility of there being an indecomposable homogeneous plane continuum which separates the plane, this paper completes the classification of chainable (by linear or circular chains) homogeneous plane continua. Bing shows that a "linearly" chainable homogeneous continuum is either a pseudo-arc or a point. Those which are "circularly" chainable and decomposable are consequently circles of pseudo-arcs or points [F. B.

Jones, same Proc. **6** (1955), 735–740; MR **17**, 180; R. H. Bing and F. B. Jones, Trans. Amer. Math. Soc. **90** (1959), 171–192; MR **20** #7251].

F. B. Jones (Chapel Hill, N.C.)

3819:

Anderson, R. D.; and Choquet, Gustave. A plane continuum no two of whose nondegenerate subcontinua are homeomorphic: An application of inverse limits. Proc. Amer. Math. Soc. **10** (1959), 347–353.

The authors show by constructive proof that there exists a plane continuum M such that (1) no subcontinuum of M separates the plane, (2) no two nondegenerate subcontinua of M are homeomorphic, and (3) M does not contain uncountably many disjoint nondegenerate subcontinua. They suggest a modification of their construction to yield a continuum M' such that (1) every nondegenerate subcontinuum of M' separates the plane, and (2) no two nondegenerate subcontinua of M' are homeomorphic. Further suggested modification of construction leads to a continuum M'' such that (1) no nondegenerate subcontinuum of M'' is imbeddable in the plane, (2) no two nondegenerate subcontinua of M'' are homeomorphic, and (3) M'' does not contain uncountably many disjoint nondegenerate subcontinua.

W. W. S. Claytor (Washington, D.C.)

3820:

Williams, R. F. Local contractions and the size of a compact metric space. Duke Math. J. **26** (1959), 277–289.

Theorems and examples concerning local contractions and local isometries are given. These deal largely with such transformations as applied to arcs, dendrites, simple closed curves and some higher dimensional manifolds, but some have setting in general metric spaces. The square of any local contraction of an arc onto itself is shown to be the identity. Also any local contraction on a dendrite with countably many endpoints is a local isometry and is pointwise periodic, whereas for unrestricted dendrites X a homeomorphism of X into X may be a local contraction but not a local isometry. A circle C may be so metrized that the rotation of C through one radian is a local contraction but not a local isometry. The effect of local contractions on Hausdorff p-dimensional measure is studied and, among other things, it is shown that such transformations do not increase this measure for any separable metric space.

G. T. Whyburn (Charlottesville, Va.)

3821:

Frolík, Zdenek. Generalisations of compact and Lindelöf spaces. Czechoslovak Math. J. **9 (84)** (1959), 172–217. (Russian. English summary)

A classificatory study of covering properties of not necessarily Hausdorff spaces. In completely regular spaces, examples are constructed of a pseudocompact space containing a non-pseudocompact zero set; a pair of countably compact subsets, P, Q, of $\beta M \times \beta M$ for any discrete space M, whose union is $\beta M \times \beta M$ and whose intersection is $M \times M$; (assuming the continuum hypothesis) an uncountable pseudocompact space having no infinite countably compact subspace. Theorem: For any space, every open covering of power m has a finite subcovering if and only if every directed set of m upper semicontinuous real functions converging pointwise monotonically down to 0 contains a sequence converging uniformly to 0. The other results constitute a fairly thorough treatment of many questions of this sort, and several of the other examples, though well known, have not been published before.

J. Isbell (Lafayette, Ind.)

3822:

Heider, L. J. Compactifications of dimension zero. Proc. Amer. Math. Soc. **10** (1959), 377–384.

Several necessary and sufficient conditions are given that the Stone-Čech compactification βX of a completely regular space X have dimension zero (i.e., have a base of open and closed sets). The main one is: βX has dimension zero if and only if the system of countable unions of open and closed sets includes every system of open sets that is completely regular in the sense of J. Kerstan [Math. Nachr. **17** (1958), 27–46; MR **20** #1968]. Some applications are given, but no new examples (or counter-examples) of zero-dimensional spaces X with βX zero-dimensional are supplied.

M. Henriksen (Lafayette, Ind.)

3823:

Banaschewski, Bernhard. On the Katětov and Stone-Čech extensions. Canad. Math. Bull. **2** (1959), 1–4.

If E is a Hausdorff space, let κE denote the maximal absolutely closed Hausdorff extension of E introduced by the reviewer [Časopis Pěst. Mat. Fys. **69** (1940), 36–49; **72** (1947), 17–32; MR **1**, 317; **9**, 153]. It is proved that, for a non-compact E, κE is never compact or (what amounts to the same, as stated in the note) that $\kappa E \neq \beta E$; this last assertion is known [M. Katětov, ibid. **72** (1947), 101–106; MR **9**, 522]. The note also contains a result concerning non-compact extensions of E for which the Stone-Weierstrass theorem holds.

M. Katětov (Prague)

3824:

Isiwata, Takesi. On stonian spaces. Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 147–176.

The author first generalizes the definition of Stonian space given by Dixmier [Summa Brasil. Math. **2** (1951), 151–182; MR **14**, 69]. Let X be a completely regular T_1-space. Then X is said to be Stonian if for every open subset U of X, every bounded real-valued continuous function defined on U can be continuously extended over all of X. The author first proves that the following properties of X are equivalent: (1) X is Stonian; (2) X is extremally disconnected (i.e., every open subset has open closure); (3) under the usual ordering, the space of all bounded continuous real-valued functions on X is a conditionally complete lattice; (4) under the usual ordering, the space of all continuous real valued functions on X is a conditionally complete lattice; (5) all proper open subspaces of X are Stonian; (6) if $X \subset Y \subset \beta X$, then Y is Stonian; (7) if U and V are open subsets of X and $U \cap V = \emptyset$, then $\bar{U} \cap \bar{V} = \emptyset$; (8) for every proper open dense subspace U of X, $\beta U = \beta X$; (9) every proper dense subspace of X is Stonian; (10) for every open subset U of X, the set of continuous real-valued functions f such that $f \geq 0$ and $f \leq \chi_U$ has a least upper bound in the lattice of all real-valued continuous functions. Spaces for which every subspace is Stonian are also discussed. The author next introduces the notion of duality. If $\beta(\beta X \cap X') = \beta X$, then X is said to admit $\beta X \cap X'$ as its dual. The author exhibits a Stonian space having a dual homeomorphic to

itself. The remainder of the paper is devoted to a study of linear functionals on the space of continuous real-valued functions on X and on the space of bounded continuous real-valued functions on X. Particular attention is paid to the representability of such functionals by integrals with respect to Baire or Borel measures on X, and the properties of these measures. In this discussion, the author relies on and borrows from results of Hewitt [Fund. Math. **37** (1950), 161–189; MR **13**, 147], Glicksberg [Duke Math. J. **19** (1952), 253–261; MR **14**, 288], and T. Ishii [Sûgaku 8 (1956/57), 153–157, 213–215; MR **20** #947, #948]. The results are too complicated to reproduce here. {Unfortunately the paper contains many misprints, some of them confusion producing.} *E. Hewitt* (Seattle, Wash.)

3825:

Isiwata, Takesi. Some properties of F-spaces. Proc. Japan Acad. **35** (1959), 71–76.

Two classes of completely regular spaces X that are investigated in L. Gillman and M. Henriksen, Trans. Amer. Math. Soc. **82** (1956), 366–391 [MR **18**, 9] are (1) the F-spaces, and (2) the spaces for which the lattice $C(X)$ is σ-complete. The present paper considers these, as well as the two subclasses whose members are defined by replacing complements of zero-sets by open F_σ's in suitable characterizations of (1) and (2). It is shown that it is equivalent to impose the defining condition for members of each class on the space X itself, or on certain subspaces of βX that contain X, or on certain proper subspaces of X. Also, for X in each of the four classes, particular kinds of quotient spaces of subspaces of βX that contain X are studied. {There are several confusing misprints. The more misleading are amended as follows: On p. 73, l. 8, read "$\overline{P(f)}$"; on p. 74, l. 18, read "$\overline{U}$" (second time); on p. 75, l. 3*, read "$k(x)=g(x)+(g(b)-g(a))h(x)$".}

C. W. Kohls (Urbana, Ill.)

3826:

Mařík, Jan. On pseudo-compact spaces. Proc. Japan Acad. **35** (1959), 120–121.

Let Z be the family of all those sequences of continuous real-valued functions on a space X which converge pointwise to zero, and $Z_0 \subset Z$ the subfamily of bounded such sequences. Let the subsets N; E; U; of Z be all those sequences which are, respectively, non-increasing; equicontinuous; uniformly convergent. Main theorem: (a) If any one of N, $N \cap Z_0$, E, $E \cap Z_0$, is contained in U, X is pseudo-compact. (b) If X is pseudocompact, then $N \cap Z_0 = N \subset U = E = E \cap Z_0$. This slightly generalizes a theorem of Iseki [same Proc. **33** (1957), 424–428; MR **20** #2682]. *J. Dugundji* (Los Angeles, Calif.)

3827:

Nagata, J. Note on dimension theory for metric spaces. Fund. Math. **45** (1958), 143–181.

Detailed proofs are given of results (mainly various conditions necessary and sufficient for $\dim R \leq n$, R metrizable, expressed in terms of sequences of coverings, of properties of metrics, and of imbedding into suitable spaces) contained in the author's previous notes [Proc. Japan Acad. **32** (1956), 166–170, 237–240, 568–573; MR **18**, 224; **19**, 156, 300; J. Inst. Polytech. Osaka City Univ. Ser. A **8** (1957), 9–14]. *M. Katětov* (Prague)

3828:

de Groot, J. Some special metrics in general topology. Colloq. Math. **6** (1958), 283–386.

In a previous paper [Canad. J. Math. **9** (1957), 511–514; MR **19**, 874], the author obtained the following theorem. A separable metric space M has dimension $\leq n$ if and only if there is a topologically equivalent, totally bounded metric d on M with the following property: for every $n+3$ points $x, y_1, y_2, \cdots, y_{n+2}$ in M, there are three indices i, j, k $(i \neq j)$ such that $d(y_i, y_j) \leq d(x, y_k)$. This result is a simplification of, and its proof makes use of, a theorem of J. Nagata [Proc. Japan Acad. **32** (1956), 237–240; MR **19**, 156], which applies also to non-separable metric spaces. As the proof of Nagata's theorem is rather complicated, the author proposes, in the present paper, the problem of finding a simpler proof for the necessity of his condition. For $n=0$, such a proof can be obtained by imbedding the 0-dimensional M into the Cantor discontinuum which admits a non-archimedean metric. For $n>0$, the author suggests a possible approach by suitably remetrizing Menger's compact universal n-dimensional space. The remaining part of the paper deals with two other problems. A topological property P is said to be finitely additive if it holds for every topological space which is the union of a finite number of closed subspaces each having property P. A short proof is given for the known fact [J. Nagata, J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **1** (1950), 93–100; MR **13**, 264; see also R. H. Bing, Duke Math. J. **14** (1947), 511–519; MR **9**, 521] that metrizability is a finitely additive property. Regularity and normality are finitely additive [S. Mrówka, Bull. Acad. Polon. Sci. Cl. III **5** (1957), 951–956; MR **20** #2678], while complete regularity is not. The author proposes the problem of inquiring systematically which important topological properties are finitely additive. The following problem is also proposed. Given a closed set S in a metrizable space M, can every metric on S (compatible with the topology of S) be extended to a metric d on M (compatible with the topology of M) such that for any $x, y \in S$, S contains all points $z \in M$ satisfying $d(x,y) = d(x,z)+d(z,y)$? It is mentioned in a footnote that this last problem has been solved in the affirmative by W. Nitka. *Ky Fan* (Notre Dame, Ind.)

3829:

Nitka, Witold. Remarks on sets convex in the sense of J. de Groot. Nederl. Akad. Wetensch. Proc. Ser. A. **62** = Indag. Math. **21** (1959), 36–38.

The following result is proved and answers a question raised by J. de Groot [see the preceding review] and E. Marczewski. Given a metrizable space M and a closed subset S of M, every metric d_S on the subspace S (compatible with the topology of S) can be extended to a metric d on M (compatible with the topology of M) with the property that $d(x,y)=d(x,z)+d(z,y)$ implies $x, y, z \in S$. *Ky Fan* (Notre Dame, Ind.)

3830:

Katětov, Miroslav. On the relation between the metric and topological dimension. Czechoslovak Math. J. 8 (**83**) (1958), 163–166. (Russian. English summary)

If P is a metric space, denote by $\mu \dim P$ (the metric dimension of P) the least r such that for every $\varepsilon > 0$ there

is an open ε-covering of P of order $\leq r+1$. Denote by dim P (the topological dimension of P) the least n such that every finite open covering of P has an open refinement of order $\leq n+1$. Sitnikov [Dokl. Akad. Nauk SSSR **88** (1953), 21–24; MR **14**, 894] has shown that for every $n \geq 3$ there is a space with dim $= n-1$ and μ dim $= [n/2]$. The author now proves that μ dim $P \leq$ dim $P \leq 2\mu$ dim P.

H. Komm (Troy, N.Y.)

3831:

Hayashi, Yoshiaki. On a dimension function. Bull. Univ. Osaka Prefecture Ser. A **6** (1958), 15–17.

The author defines a dimension function $D(X)$ for separable metric spaces X by the condition $D(X) \leq n$ if X contains no subset homeomorphic to $(n+1)$-dimensional Euclidean space R^{n+1}. He asserts that $D(X \times Y) = D(X) + D(Y)$, but his proof has a serious gap. {R. Rosen [Thesis, Univ. of Wisconsin, 1959] has constructed a separable metric space X which contains no subset homeomorphic to R^3 but $X \times R^1$ is homeomorphic to R^4.}

E. Dyer (Chicago, Ill.)

3832:

Weston, J. D. Incomplete subspaces in a Hausdorff space. Arch. Math. **10** (1959), 40–41.

Call a space t-complete provided it is complete under some metric which determines its topology. Aided by one of his results on the comparison of topologies [J. London Math. Soc. **32** (1957), 342–354; MR **20** #1288], the author proves the following: Suppose a Hausdorff space X is a union $\bigcup_{a \in A} S_a$ of at least two pairwise disjoint homeomorphic subspaces S_a, each of which is dense in X. Then no space S_a is t-complete. From this he deduces the reviewer's result asserting that a t-complete topological group must be complete under every two-sided invariant metric which determines its topology [Proc. Amer. Math. Soc. **3** (1952), 484–487; MR **13**, 848]. {Reviewer's remarks: (1) Of course a t-complete topological group need not admit any two-sided invariant metric which determines its topology—consider, for example, the group of all homeomorphisms of $[0, 1]$ onto itself. However, the author's theorem implies further that if a topological group is t-complete, then it cannot be obtained as a dense proper subgroup of any topological group, and hence must be complete under its bilateral uniform structure. [See Ex. 6 on pp. 31–32 of Chap. 3 (Groupes Topologiques) of Bourbaki's *Topologie générale*, Actualités Sci. Ind. no. 916, Hermann, Paris, 1942; MR **5**, 102.] (2) When X is metrizable, the author's theorem can be sharpened as follows: Suppose S_1 and S_2 are disjoint dense subsets of X. Then if both are t-complete, they are both absolute G_δ sets relative to metric spaces and hence are dense G_δ subsets of the metric completion of X, whence their intersection is dense in X, an impossibility. Thus S_1 and S_2 cannot both be t-complete.}

V. L. Klee, Jr. (Copenhagen)

3833:

Dolcher, Mario. Determinazione del massimo numero di eccezioni alla n-vocità dell'inversa di una trasformazione piana continua. Ann. Univ. Ferrara. Sez. VII (N.S.) **7** (1957/58), 53–64. (French summary)

A disc with oriented boundary C is mapped into E_2 by ϕ. Let A be a component of $E_2 - \phi C$ and let n be the index of points of A relative to ϕC. A point p of A is exceptional if the cardinality $\nu(p)$ of $\phi^{-1}(p)$ is less than n. It is shown that if $p_1, \cdots, p_k$ are exceptional, then $\sum (n - \nu(p_i)) \leq n-1$. There exist examples in which the equality holds.

P. A. Smith (New York, N.Y.)

ALGEBRAIC TOPOLOGY

See also 3347, 3426, 3806, 4018.

3834:

Lee, Ke-chun. Kombinatorische Invarianten von endlichem Komplex. Sci. Sinica **8** (1959), 449–460.

German version of article below.

3835:

Lee, Ke-chun. Kombinatorische Invarianten von endlichem Komplex. Acta Math. Sinica **8** (1958), 473–482. (Chinese. German summary)

Let K be an n-dimensional finite simplicial complex and let $a_{ij,k}(K)$ denote the number of ordered pairs (σ_i, σ_j) of an i-dimensional simplex σ_i and a j-dimensional simplex σ_j in K which have a k-dimensional face in common. In this note, the author proves that every combinatory invariant of K of the form

$$\theta(K) = \sum_{\substack{i,j=0 \\ k \geq -1}}^{n} \alpha_{ij,k} a_{ij,k}(K)$$

with real coefficients $\alpha_{ij,k}$ can always be written in the form

$$\theta(K) = \alpha\chi(K) + \beta\chi^2(K) + \gamma\chi^+(K),$$

where α, β, γ are real numbers and

$$\chi(K) = \sum_{i=0}^{n} (-1)^i a_{ii,i}(K),$$

$$\chi^2(K) = \sum_{\substack{i,j=0 \\ k \geq -1}}^{n} (-1)^{i+j} a_{ij,k}(K),$$

$$\chi^+(K) = \sum_{i,j=0}^{n} (-1)^{i+j} a_{ij,-1}(K).$$

Sze-tsen Hu (Palo Alto, Calif.)

3836:

Leslie, Joshua. Modules simpliciaux sur une algèbre simpliciale. C. R. Acad. Sci. Paris **248** (1959), 2692–2694.

The author defines, in a natural manner, a c.s.s. algebra (complete semisimplicial algebra) over the integers, and the operation of such an algebra A on a c.s.s. module; A-modules form an abelian category $\mathfrak{M}$.

If M is an abelian group and X a c.s.s. complex, the abelian group of maps $X_n \to M$ is denoted by $C^n(X, M)$. By Δ_q denote the standard c.s.s. q-simplex. Define the c.s.s. group $C^n(M)$ by putting $(C^n(M))_q = C^n(\Delta_q, M)$, the face and degeneracy operators being transpositions of the familiar maps $\Delta_{q-1} \to \Delta_q$, $\Delta_q \to \Delta_{q+1}$, respectively. In a well-known manner there is a natural isomorphism $C^n(X, M) \approx \operatorname{Hom}(X, C^n(M))$, the group of c.s.s. maps $X \to C^n(M)$.

Let A be a c.s.s. algebra and $\mathfrak{M}_n$ the category of A_n-modules. Using the above natural isomorphism,

$C^n(M)$ for $M \in \mathfrak{M}_n$ can be given the structure of an A-module; i.e., if $M \in \mathfrak{M}_n$, then $C^n(M) \in \mathfrak{M}$.

If $I \in \mathfrak{M}_n$ is injective (in $\mathfrak{M}_n$!), then $C^n(I)$ is injective in $\mathfrak{M}$. By using the many injectives thus obtained, and the method of Eilenberg-Cartan-Grothendieck, the author proves the theorem: Every object of $\mathfrak{M}$ can be (naturally) embedded in an injective object.

V. Gugenheim (Baltimore, Md.)

3837a:

Shih, Weishu. Sur les groupes simpliciaux abéliens et le théorème des coefficients universels. C. R. Acad. Sci. Paris **247** (1958), 2079–2081.

3837b:

Shih, Weishu. Sur les groupes simpliciaux abéliens et le théorème des coefficients universels. C. R. Acad. Sci. Paris **248** (1959), 346.

The author points out in the second of these two notes that all of the new results supposedly proved in the first are false.

J. C. Moore (Princeton, N.J.)

3838:

Darbo, Gabriele. Teoria dell'omologia in una categoria di mappe plurivalenti ponderate. Rend. Sem. Mat. Univ. Padova **28** (1958), 188–220.

The purpose of this paper is to construct a homology theory in a category of (Hausdorff) spaces and weighted maps, which are essentially transformations which carry points into linear combinations of points with coefficients in a ring. (Certain reasonable conditions are imposed.) The resulting theory satisfies Eilenberg-Steenrod axioms, suitably reinterpreted, plus a linearity axiom for induced maps.

D. W. Kahn (Cambridge, Mass.)

3839:

Tihomirova, E. S. Some homology invariants of equimorphic transformations. Dokl. Akad. Nauk SSSR **120** (1958), 475–476. (Russian)

A map of one metric space onto another is equimorphic if it is 1-1 and uniformly continuous both ways. A geodesic space is a complete metric space R with the property that for every two points $x, y \in R$ there is a $z \in R$ such that $\rho(x, z) = \rho(y, z) = \frac{1}{2}\rho(x, y)$ (Riemannian manifolds are geodesic spaces). If R is a geodesic space, assign to every natural number k a domain $P_k \subset R$ as follows: $x \in R$ belongs to P_k if and only if there is a singular bounding cycle Z of R whose carrier contains x such that $d(X)/[d(Z)+1] > k$ for an arbitrary chain X with boundary Z ($d(Y)$ is the diameter of the carrier of the singular chain Y). If H_k^r is the subgroup of the group of singular r-dimensional homologies of P_k consisting of the classes homologous to zero in R, the relations $P_k \supset P_{k+1}$ induce natural homomorphisms $\varphi_k : H_{k+1}^r \to H_k^r$. The limit group Q^r of this inverse system is invariant under equimorphic maps of R.

H. Komm (Troy, N.Y.)

3840:

Dauker, K. H. [Dowker, C. H.] The excision theorem. Dokl. Akad. Nauk SSSR **125** (1959), 1190–1192. (Russian)

The strong excision property for Čech cohomology, $H^q(X, A) = H^q(X-U, A-U)$ if U is an open set of X contained in A, is known for paracompact spaces [A. D. Wallace, Duke Math. J. **19** (1952), 177–182; MR **13**, 765]; and Wallace conjectured that it is false for normal spaces. The author shows that the property holds if $A-U$ is paracompact and has a collectionwise normal relative neighborhood either in A or in $X-U$, but the latter condition cannot be omitted even if X is normal.

Note that Wallace gives a stronger theorem of this type for paracompact spaces, as does Shozo Sakai [Sci. Rep. Tokyo Bunrika Daigaku. Sect. A **4** (1953), 290–297; MR **15**, 52].

J. Isbell (Lafayette, Ind.)

3841:

Luft, Erhard. Eine Verallgemeinerung der Čechschen Homologietheorie. Bonn. Math. Schr. no. 8, iii+41 pp. (1959).

This paper introduces (for paracompact spaces) a homology theory, which is a pendant to the cohomology theory with coefficients in sheaves. The starting point is the fact that cohomology groups are actually defined by means of the presheaf of sections of the given sheaf, and the notion of presheaf admits a dual notion, that of an "inverse group family" (i.g.f.). An i.g.f. $\mathfrak{G}$ on a space X is a covariant functor, defined on the category of all open sets $U \subset X$ and all inclusion maps $U \to V$, $U \subset V$; the functor takes its values in the category of abelian groups and homomorphisms.

A homology theory on X consists of a function assigning an abelian group $H_q(X, \mathfrak{G})$ to every integer q and every i.g.f. $\mathfrak{G}$; furthermore, it assigns homomorphisms of homology groups to homomorphisms $\mathfrak{G} \to \mathfrak{H}$; and finally, there is a boundary homomorphism assigned to each exact sequence

$$0 \to \mathfrak{G}' \to \mathfrak{G} \to \mathfrak{G}'' \to 0.$$

Six axioms are postulated; these are natural analogues of Cartan's axioms for cohomology [Séminaire H. Cartan, 1950/51, *Cohomologie des groupes, suite spectrale, faisceaux*, 2e éd. Secrétariat mathématique, Paris, 1955; MR **17**, 1117; exposé 16], with the exception of the exactness axiom. This axiom is assumed valid only under the additional hypothesis that a certain i.g.f. is "fine". Next, it is proved that this homology theory is unique. The proof depends on the lemma that each i.g.f. $\mathfrak{G}$ admits a "fine" i.g.f. $\mathfrak{F}(\mathfrak{G})$ and a homomorphism $\mathfrak{F}(\mathfrak{G}) \to \mathfrak{G}$ onto $\mathfrak{G}$. Finally, the theory is effectively constructed following the Čech covering procedure.

S. Mardešić (Zagreb)

3842:

Bauer, Friedrich-Wilhelm. Spezielle Homologiestrukturen. Math. Ann. **136** (1958), 348–364.

This paper is concerned with applications of the rather abstract theory developed by the author in a previous paper [same Ann. **135** (1958), 93–114; MR **20** #4262].

The first application is to the homology theory defined by K. Sitnikov for an arbitrary metric space X [Mat. Sb. (N.S.) **3476** (1954), 3–54; MR **16**, 736; p. 34]. The author discusses the Sitnikov homology theory in the framework of his general theory and proves the following generalized form of the Alexander duality theorem for any subset X of Euclidean n-space, R^n: $H_q(X) \approx H^p(R^n - X)$, where $p+q=n-1$. Here the cohomology theory used is the Čech theory based on infinite, star-finite coverings. The second

application is to the proof of a simplified form of the Eilenberg-Steenrod uniqueness theorem for the homology theory of polyhedra. The third application is to the proof of the second duality theorem of Sitnikov [Dokl. Akad. Nauk SSSR **96** (1954), 925–928; MR **17**, 70]. This theorem asserts the existence of a cohomology theory such that the generalized Alexander duality theorem mentioned above is true with the Sitnikov homology theory replaced by the Vietoris theory. The last application is to the singular homology theory. *W. S. Massey* (Providence, R.I.)

3843:

Čogošvili, G. S. Generalization of the Steenrod duality rule. Soobšč. Akad. Nauk Gruzin. SSR **21** (1958), 641–648. (Russian)

Steenrod duality [Ann. of Math. (2) **41** (1940), 833–851; MR **2**, 73] is extended to an arbitrary subset A of an n-dimensional cell Σ in the following sense: Let F_a be the directed set of compact subsets of A, quasi-ordered by inclusion, and $H_r^m(F_a)$ the homology group of regular m-cycles of F_a. The direct limit $H_r^m(A)$ of the direct system $\{H_r^m(F_a), i_{ab}\}$, where i_{ab} is induced by the inclusion $F_a \subset F_b$, is called the homology group of regular m-cycles with compact supports of A. Let $G_a = \Sigma - F_a$, and $H_\Delta{}^m(G_a)$ be the homology group of m-dimensional infinite cycles of a triangulation of the open set G_a. The set G_a is a directed set, quasi-ordered by $G_a < G_b \Leftrightarrow G_b \subset G_a$. The author defines a homomorphism $\pi_{ab}: H_\Delta{}^m(G_a) \to G_\Delta{}^m(G_b)$ for $G_a < G_b$ such that $\{H_\Delta{}^m(G_a), \pi_{ab}\}$ is a direct system. Let $H_\Delta{}^m(B)$ be the direct limit. It is explained in which sense $H_\Delta{}^m(B)$ has to be regarded as homology groups of the space $B = \Sigma - A$, using previous work of the author [Soobšč. Akad. Nauk Gruzin SSR **19** (1957), 513–520; MR **20** #2701], and it is shown that $H_r^m(A)$ and $H_\Delta{}^m(B)$ are isomorphic. The proof uses Steenrod's result

$$s_a: H_\Delta{}^m(G_a) \cong H_r^m(F_a),$$

and the fact that $i_{ab} \circ s_a = s_b \circ \pi_{ab}$.

M. A. Kervaire (New York, N.Y.)

3844:

Zeeman, E. C. A note on a theorem of Armand Borel. Proc. Cambridge Philos. Soc. **54** (1958), 396–398.

Suppose that K is a field. Let $\{E_r\}$ be a spectral sequence, where r ranges over integers greater than or equal to two. We assume E_r is bigraded positively, i.e. $E_r = \sum_{p,q \geq 0} E_r{}^{p,q}$, and that in E_r there is given a differential operator $d_r: E_r{}^{p,q} \to E_r{}^{p+r,q-r+1}$ such that the homology of E_r with respect to d_r is E_{r+1}. Let B be the graded vector space over K such that $B^p = E_2{}^{p,0}$, and F the one such that $F^q = E_2{}^{0,q}$. The spectral sequence is a canonical spectral sequence of algebras if the following conditions hold: (1) E_r is an algebra such that $E_r{}^{p,q} \cdot E_r{}^{p',q'} \subset E_r{}^{p+p',q+q'}$; (2) the natural map φ: $B \otimes F \to E_2$ is an isomorphism of algebras; (3) $d_r(xy) = d_r(x)y + (-1)^n x d_r(y)$ for $x \in E_r{}^{p,q}$, $p+q=n$, and $y \in E_r$ for all $r \geq 2$; and (4) if $x \in E_r{}^{p,q}$, $y \in E_r{}^{p',q'}$, then $xy = (-1)^{(p+q)(p'+q')} yx$. The spectral sequence is said to be trivial if $E_\infty{}^{p,q} = 0$ for $p, q \neq 0$ and $E_\infty{}^{0,0} = K$.

One of the main theorems which A. Borel proved in his thesis [Ann. of Math. (2) **57** (1953), 115–207; MR **14**, 490] may now be stated. Theorem: If $\{E_r\}$ is a trivial canonical spectral sequence of algebras, and F is an exterior algebra generated by odd dimensional elements, then: (I) generators of F, $y_1, \cdots, y_m$, may be chosen so that each y_i is transgressive and odd dimensional; and (II) $B = K[z_1, \cdots, z_m]$ is a polynomial algebra over K when z_i is an image of y_i under transgression for $i = 1, \cdots, m$.

In this note the author gives a proof of (II) assuming (I) in addition to the other hypotheses of the theorem.

J. C. Moore (Princeton, N.J.)

3845:

★Hirzebruch, F.; und Scheja, G. Garben- und Cohomologietheorie. Ausarbeitungen mathematischer und physikalischer Vorlesungen, Bd. 20. Aschendorff, Münster, 1957. iv + 236 pp. DM 20.00.

This is a standard exposition, in some detail, of the cohomology theory of sheaves. The contents of the chapters are as follows: I. Topological foundations; II. Algebraic foundations; III. General sheaf theory; IV. Cohomology of sheaves. *M. F. Atiyah* (Cambridge, England)

3846:

Watts, Charles E. On phi-families. Proc. Amer. Math. Soc. **10** (1959), 369–371.

A functor T is defined, analogous to 1-point compactification, which assigns to each pair $(X, \mathscr{F})$, where $\mathscr{F}$ is a phi-family of supports on the space X [in the sense of Séminaire H. Cartan, 1950/1951, Secrétariat mathématique, Paris, 1955; MR **17**, 1117], a space X^*, and to each sheaf on X a sheaf on X^*. If H is the cohomology functor, then $HT \cong H$; and so the notion of phi-family is rendered theoretically redundant.

E. C. Zeeman (Cambridge, England)

3847:

Curtis, M. L.; and Fort, M. K., Jr. The fundamental group of one-dimensional spaces. Proc. Amer. Math. Soc. **10** (1959), 140–148.

The authors continue their study of a one-dimensional separable metric space X begun in their earlier paper, Proc. Amer. Math. Soc. **8** (1957), 577–579 [MR **19**, 158]. There it was shown that $\pi_n(X) = 0$ if $n > 1$, and now $\pi_1(X)$ is studied in detail. In the plane, let S be the bouquet of coaxial circles $(x - 1/n)^2 + y^2 = 1/n^2$ $(n = 1, 2, \cdots)$, and suppose X is a locally connected continuum. Theorem 1: If $Y \subset X$ and $y \in Y$, $\pi_1(Y, y)$ is a subgroup of $\pi_1(X, y)$. Theorem 2: Either (a) X is locally simply connected and $\pi_1(X)$ is finitely generated and free, or (b) X contains a subset of the homotopy type of S. If (b), then by 1, $\pi_1(X)$ contains a copy of $\pi_1(S)$, which is the "topologists' product" investigated by the reviewer in Proc. London Math. Soc. (3) **6** (1956), 455–480 [MR **18**, 192]; thus $\pi_1(X)$ is uncountable and not free, even though, as the authors show, $\pi_1(X)$ never contains elements of finite order. Further results are obtained, concerning the abelian subgroups of $\pi_1(X)$. *H. B. Griffiths* (Bristol)

3848:

Curtis, M. L.; and Fort, M. K., Jr. Singular homology of one-dimensional spaces. Ann. of Math. (2) **69** (1959), 309–313.

The authors continue their investigation of one-dimensional separable metric spaces [#3847 above]. In this paper, they study singular homology and prove that, for such a space X, $H_k(X) = 0$, $k > 1$.

The proof, briefly, runs as follows: by a previous result, $\pi_k(X)=0$, $k>1$. Hence X is an Eilenberg-MacLane space of type $(\pi, 1)$, where $\pi=\pi_1(X)$ and $H_k(X)=H_k(\pi, 1)$. Using the facts that (1) the functor $H_k(\cdot, 1)$ commutes with direct limits, (2) an inverse limit of free groups of finite rank is locally free (its finitely generated subgroups are free), it follows that $H_k(\pi, 1)=0$, $k>1$, if π is such an inverse limit. Next, the fundamental group of the Menger universal one-dimensional curve satisfies the latter condition. This, together with the fact that any one-dimensional separable metric space is imbeddable in the Menger curve, gives the result.

J.-P. Meyer (Baltimore, Md.)

3849:

Crowell, Richard H. On the van Kampen theorem. Pacific J. Math. **9** (1959), 43–50.

Following R. H. Fox [mimeographed notes, 1955], the author gives a definition of the direct limit of a system of groups and homomorphisms, together with a procedure for deriving a presentation of the limit from presentations of the groups in the system. In view of this, the following constitutes a generalization of the van Kampen theorem. Let a topological space X be the union of a family of pathwise connected open sets X_α, closed under finite intersection, and with all X_α containing a common point p; then the fundamental groups of the X_α at p, together with the homomorphisms induced by inclusion, constitute a system whose limit is the fundamental group of X at p.

R. C. Lyndon (Princeton, N.J.)

3850:

Spanier, E. H.; and Whitehead, J. H. C. Duality in relative homotopy theory. Ann. of Math. (2) **67** (1958), 203–238.

As the authors say in the introduction to this paper, it extends the duality they introduced earlier [E. H. Spanier and J. H. C. Whitehead, Mathematica **2** (1955), 56–80; MR **17**, 653]. In particular the duality of finite polyhedra to the category of finite polyhedral lattices and S maps between them restricted by carriers. One of the most interesting results is obtained in § 11: the S-homotopy exact couple of a finite polyhedral lattice X is isomorphic with the S-cohomotopy exact couple of X^*, where X^* is a combinatorial n-dual of X.

J. C. Moore (Princeton, N.J.)

3851:

Spanier, E. Infinite symmetric products, function spaces, and duality. Ann. of Math. (2) **69** (1959), 142–198; erratum, 733.

Part I is devoted primarily to a proof of the Dold-Thom theorem. There is also a careful discussion of the weak topology relative to compact subsets in a space.

The main results are in Part II. Let $\mathrm{SP}^\infty X$ denote the infinite symmetric product of X, let $[X, Y]$ denote the set of homotopy classes of maps of X into Y and let $\{X, Y\}$ denote the set of S-homotopy classes of maps of X into Y. If X and Y admit multiplications (precisely: are weak abelian monoids) let $[X, Y]_H$ be the set of H-homotopy classes of maps of X into Y, where H-homotopy means homotopy through homomorphisms. Ordinarily there is a map $[X, Y]\to[\mathrm{SP}^\infty X, \mathrm{SP}^\infty Y]_H$. The author extends this to a map $\mathrm{SP}:\{X, Y\}\to[\mathrm{SP}^\infty X, \mathrm{SP}^\infty Y]_H$ and shows that for any $\phi\in\{X, X'\}$ there is a commutative diagram

$$\begin{tikzcd} H_q(X) \arrow[r, "\phi_*"] \arrow[d, "\tau"] & H_q(X') \arrow[d, "\tau"] \\ \pi_q(\mathrm{SP}^\infty X) \arrow[r, "(\mathrm{SP}\phi)_\sharp"] & \pi_q(\mathrm{SP}^\infty X') \end{tikzcd}$$

where τ is the isomorphism of the Dold-Thom theorem. Thus SP converts homology to homotopy.

Next he reviews the duality in S-theory and recalls in particular that if X, Y are subpolyhedra of S^{n+1} and X^*, Y^* are $(n+1)$-duals of X, Y respectively, then there is an isomorphism $D_{n+1}:\{X, Y\}\approx\{Y^*, X^*\}$ and a commutative diagram

$$\begin{tikzcd} H_q(X^*) \arrow[r, "\theta"] & H^{n-q}(X) \\ H_q(Y^*) \arrow[r, "\theta"] \arrow[u, "(D_{n+1}\phi)_*"] & H^{n-q}(Y) \arrow[u, "\phi^*"'] \end{tikzcd}$$

where θ is an isomorphism (essentially that of the Alexander duality theorem). Thus D_{n+1} converts cohomology into homology.

Finally he notes that there are natural maps

$$\mathrm{SP}^\infty S^n \xrightarrow{\rho} \Omega\,\mathrm{SP}^\infty S^{n+1} \to \cdots \to \Omega^k\,\mathrm{SP}^\infty S^{n+k} \xrightarrow{\Omega^k\rho} \cdots$$

(Ω as usual is the loop-functor) and each of the spaces above is a $K(Z, n)$ space. Let $F_{n,k}(X)$ be the space of maps $X\to\Omega^k\,\mathrm{SP}^\infty S^{n+k}$ with the compact-open topology. There are induced maps

$$F_{n,0}(X)\to F_{n,1}(X)\to\cdots\to F_{n,k}(X)\to\cdots$$

which are all proved to be imbeddings. Define $F_n(X)=\lim_k F_{n,k}(X)$. F_n is a functor $[X, X']\to[F_nX', F_nX]_H$. It can be extended to a functor $F_n':\{X, X'\}\to[F_nX', F_nX]_H$. There is a commutative diagram for any $\phi\in\{X, X'\}$:

$$\begin{tikzcd} H^{n-q}(X') \arrow[r, "\phi^*"] & H^{n-q}(X) \\ \pi_q(F_nX') \arrow[r, "(F_n'\phi)_\sharp"] \arrow[u, "\Delta_n"] & \pi_q(F_nX) \arrow[u, "\Delta_n"'] \end{tikzcd}$$

where Δ_n is an isomorphism due essentially to Federer [Trans. Amer. Math. Soc. **82** (1956), 340–361; MR **18**, 59] and Thom [Colloq. Topologie Algébrique, Louvain, 1956, pp. 29–39, Thone, Liège, 1957; MR **19**, 669]. Thus F_n' converts cohomology to homotopy.

Now the author exhibits a natural map N which makes the following diagram commutative:

$$\begin{tikzcd} \{X_1, X_2\} \arrow[r, "D_{n+1}"] \arrow[d, "F_n'"] & \{X_2^*, X_1^*\} \arrow[d, "\mathrm{SP}"] \\ {[F_n'X_2, F_n'X_1]_H} \arrow[r, "N"] & {[\mathrm{SP}^\infty X_2^*, \mathrm{SP}^\infty X_1^*]_H} \end{tikzcd}$$

This proves his main result that there is a natural weak homotopy equivalence between $\mathrm{SP}^\infty D_{n+1}(X)$ and $F_n(X)$.

N. Stein (New York, N.Y.)

3852:

Jensen, Anton. On the homotopy structure of coverings. Math. Scand. 6 (1958), 137–159.

Let α and β be coverings of topological spaces X and Y each indexed by a fixed set J. A map of α into β is a continuous map F of X into Y such that for each $\nu \in J$, $f(A_\nu) \subset B_\nu$, $A_\nu \in \alpha$, $B_\nu \in \beta$. Homotopy between maps and homotopy equivalence of coverings is defined in the obvious way. This paper extends the definitions of singular complex, minimal subcomplex and Postnikov system of a topological space to coverings in a way consistent with the above notion of homotopy equivalence of coverings. For example, the singular complex of a covering α is a C.S.S. complex whose n-simplexes have the form

$$((m_0, m_1, \cdots, m_n), T)$$

where m_k is a finite subset of J, $m_k \subset m_{k+1}$, $T \in S(X)_n$ and $m_k \subset \{\nu \in J \mid \text{image } \partial_0{}^k T \subset A_\nu \in \alpha\}$. In defining minimal subcomplex one considers deformations of T which preserve the inclusion relations prescribed by

$$(m_0, m_1, \cdots, m_n).$$

E. H. Brown (Waltham, Mass.)

3853:

James, I. M. Whitehead products and vector-fields on spheres. Proc. Cambridge Philos. Soc. 53 (1957), 817–820.

By an r-field on the n-sphere, S^n, is meant a continuous function which assigns to each point of S^n an r-frame in the tangent space at that point. In the first part of this note, the author points out the following three corollaries to the main results of one of his recent papers [Proc. London Math. Soc. (3) 8 (1958), 536–547; MR 20 #7268]: (1) If S^n admits an r-field, where $r<n$, then S^{2n+1} admits an $(r+1)$-field. (2) Let q denote the least value of n such that S^n admits an r-field. Then $q+1$ is a power of 2. (3) Let m and n be $\geq r$. If S^m and S^n admit r-fields, then so does S^{m+n+1}. Conversely, if S^n and S^{m+n+1} admit r-fields, and if $m>2r$, then S^m admits an r-field.

Let G_q denote the stable homotopy group of the q-stem, which is isomorphic to $\pi_{n+q}(S^n)$ for $n>q+1$. The second part of this note contains a proof of the following conjecture of Serre: There exists an infinity of values of q such that G_q contains an element of even order.

Let $w_n \in \pi_{2n-1}(S^n)$ denote the Whitehead product of a generator of $\pi_n(S^n)$ with itself. The third part of this note contains the proof of the following theorem: If S^n admits an r-field $(r \leq n)$, then w_n is an r-fold suspension. Conversely, if w_n is an r-fold suspension and $n>2r$, then S^n admits an r-field. Several corollaries are deduced from this theorem. *W. S. Massey* (Providence, R.I.)

3854:

Chang, Su-cheng. On intrinsic inequalities associated with certain continuous mappings and their application to fiber spaces. Acta Math. Sinica 9 (1959), 51–68. (Chinese. English summary)

This paper contains the details of an earlier note of the author. See Sci. Record (N.S.) 2 (1958), 98–100 [MR 20 #5476]. *Sze-tsen Hu* (Detroit, Mich.)

3855:

Dold, Albrecht. Die geometrische Realisierung eines schiefen kartesischen Produktes. Arch. Math. 9 (1958), 275–286.

Intuitively a twisted Cartesian product of two semi-simplicial complexes F and B is almost a fibre space with projection map $\pi: E \rightarrow B$ and fibre F, where E is the total complex of the twisted Cartesian product. In this paper the author proves three main theorems.

Theorem 1: Let (F, B, E) be a twisted Cartesian product which has a structural monoid complex H, such that $\pi_0(H)$ (the path components of H) form a group; then the geometric realization of the map $\pi: E \rightarrow B$ is a quasi-fibration, with total space the geometric realization of E, base the realization of B, and fibre the realization of F.

Theorem 2: Let K be a semi-simplicial complex with base point, A a subcomplex with the same base point, and K/A the complex obtained by collapsing A to a point. Let SP(K), SP(K/A), and SP(A) denote the infinite symmetric products of the complexes K, K/A, and A. Under these conditions there exists a twisted Cartesian product (SP(A), SP(K/A), SP(K)) with fibre SP(A), base SP(K/A), and total complex SP(K). This twisted Cartesian product is principal with structural monoid SP(A).

Theorem 3: If Γ is an associative H-space with unit such that $\pi_0(\Gamma)$ is a group, then Γ has the same weak homotopy type as the space of loops in a connected space.

The first of these theorems shows that one's intuition is correct: a twisted Cartesian product is almost a fibre space. The second is useful in abstracting the earlier work of the author and R. Thom on infinite symmetric products in the geometric case [A. Dold and R. Thom, Ann. of Math. (2) 67 (1958), 239–281; MR 20 #3542] to the semi-simplicial case. The third theorem is a further application of the first theorem. *J. C. Moore* (Princeton, N.J.)

3856:

Hu, Sze-Tsen. On fiberings with singularities. Michigan Math. J. 6 (1959), 131–149.

Let E, B be spaces, let A be a subspace of B and let $p: E \rightarrow B$ be a map. Then p is called a fibering of E over B with A as singularities if it is onto B and if $p \mid p^{-1}(B-A)$ is a Serre fibre map onto $B-A$. The regular fibres of p are the sets $p^{-1}(b)$, $b \in B-A$, and the singular fibres of p are the sets $p^{-1}(a)$, $a \in A$. If A is closed in B one may show that p factorizes as p_1p_2 where the singular fibres of p_1 and the regular fibres of p_2 are single points; indeed p_2 just identifies each singular fibre of p to a point. Following this remark the author concentrates attention on the case when p has only one singular fibre which is a single point, and uses the notion as an approach to the study of the local invariants of a path-connected space X at a point x_0.

The author has defined the tangent space $T(X, x_0)$ of X at x_0 to be the space of paths on X emanating from x_0 and never returning. Then the projection $p: T \rightarrow X$ given by $p(l) = l(1)$ is a fibre map of T onto $X - x_0$. If we adjoin the constant path, e_0, at x_0 to T to obtain $E(X, x_0)$, then $p: E \rightarrow X$ is a fibre map with one punctual singular fibre, and the local (homology and homotopy) invariants of E at e_0 are shown to be isomorphic to those of X at x_0, that is (using the author's definition of local invariants), to the global invariants of T.

If $f: X, x_0 \rightarrow Y, y_0$ is a fibre map with x_0 as its one singular fibre, then f induces $f: E(X, x_0) \rightarrow E(Y, y_0)$ and if f is also a fibre map with e_0 as its one singular fibre then the usual apparatus of fibre-space theory may be applied to $f \mid T(X, x_0)$ to yield relations between the local invariants

of X at x_0, the local invariants of Y at y_0 and the global invariants of the fibre of f. The author describes one convergence-type condition on f which ensures that f is a fibre map.
P. J. Hilton (Birmingham)

3857:

Lee, Pei-shing. Characteristic classes on local coefficients. Acta Math. Sinica **8** (1958), 384–395. (Chinese. English summary)

Let B be an $(m-1)$-sphere bundle over a finite polyhedron with the orthogonal group O_m as structural group. It is known that the Stiefel-Whitney classes $W^q(B)$ reduced mod 2 are characteristic classes in the sense of Pontrjagin. In this paper, the author generalizes Pontrjagin's method to local coefficients and proves that the Stiefel-Whitney classes $W^q(B)$, where q is odd or $q=m$, are Pontrjagin characteristic classes in this generalized sense.
Sze-tsen Hu (Palo Alto, Calif.)

3858:

Frum-Ketkov, R. L. The behavior of cycles not homologous to zero under the mapping of an n-manifold into n-dimensional Euclidean space. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 42–44. (Russian)

Let M^n, $M_1{}^n$ be closed orientable n-dim. manifolds, R^n n-dim. Euclidean space, $H_s(K)$ the s-dim. rational homology group of K, and $p^s(K)$ the rank of this group. Let $f\colon M^n \to R^n$ or $f\colon M^n \to M_1{}^n$ be a continuous map, and ζ^s a non-zero homology class of M^n. Say that cycles from this homology class are 'mapped null' if in R^n or $M_1{}^n$ there is a polyhedron L, dim $L \leqq s$, such that for any $\varepsilon>0$ the set $f^{-1}(\overline{O(L, \varepsilon)})$ contains cycles from the class, and the images of these cycles are homologous to zero in $\overline{O(L, \varepsilon)}$. Define: $r_f{}^s$ = maximal number of independent s-dim. classes containing cycles which are mapped null; $q_f{}^s$ = maximal number of independent s-dim. classes of $f(M^n)$; $p_f{}^s$ = rank of the image of $H_s(M^n)$ in $H_s(R^n)$ or $H_s(M_1{}^n)$. Theorem 1: Let $f\colon M^n \to M_1{}^n$ or $f\colon M^n \to R^n$. Suppose $p^s(M^n) \neq 0$, and that deg $(f)=0$. Then

$$q_f{}^s + q_f{}^{n-s} \leqq \tfrac{1}{2}(p^s(M^n) + p^{n-s}(M^n));$$
$$q_f{}^s \leqq r_f{}^{n-s}; \quad q_f{}^{n-s} \leqq r_f{}^s.$$

Let K be a polyhedron, $f\colon M^n \to K$. Denote by $\mu_f{}^s$ the maximal number of independent classes of $H_s(M^n)$ such that each class contains representatives whose images in K do not contain non-null s-dim. cycles. Denote by $\bar{\mu}_f{}^s$ the maximal number of independent classes such that for $\varepsilon>0$ there exist representatives such that no ε-translation of images of these representatives contains a non-null s-dim. class. Now, assume $M_1{}^n$ is triangulated. Theorem 2: Let f be a continuous map $M^n \to R^n$ or $M^n \to M_1{}^n$. Suppose $p^s(M^n) \neq 0$. Then $\mu_f{}^s + \bar{\mu}_f{}^{n-s} \geqq p^s(M^n)$. Theorem 3: If $p^1(M^n) \neq 0$, then $\mu_f{}^1 + \mu_f{}^{n-1} \geqq p^1(M^n)$.

Theorem 1 generalizes a theorem of H. Hopf [J. Reine Angew. Math. **163** (1930), 71–88]. The author outlines proofs for special cases of the first two theorems.
D. W. Kahn (New Haven, Conn.)

3859:

★**Alexandroff, P. Die topologischen Dualitätssätze. I. Abgeschlossene Mengen.** Mathematische Forschungsberichte, VII. VEB Deutscher Verlag der Wissenschaften, Berlin, 1959. 106 pp. Brosch.: DM 16.80.

Translation from the Russian [Trudy Mat. Inst. Steklov. no. 48, Izdat. Akad. Nauk SSSR, Moscow, 1955; MR **17**, 1118].

3860:

Conner, P. E.; and Floyd, E. E. On the construction of periodic maps without fixed points. Proc. Amer. Math. Soc. **10** (1959), 354–360.

Let r be an integer greater than one and not the power of a prime. Let T be a transformation $X \to X$ of period r. It has been questioned whether or not T necessarily admits fixed points when X is a euclidean space. The authors construct remarkable examples suggesting strongly that the answer is in the negative. In the first example X is a contractible star finite 4-dimensional simplicial complex on which T acts simplicially and without fixed points. In the second T acts without fixed points on a contractible manifold M of dimension $>4r$. There is some reason to think that M may be homeomorphic to a euclidean space.
P. A. Smith (New York, N.Y.)

3861:

Bourgin, D. G. Deformation and mapping theorems. Fund. Math. **46** (1959), 285–303.

L'auteur démontre des théorèmes, et quelques applications, concernant les images de familles de k-uples de points d'une sphère S^n par une application continue, ou par une famille à un paramètre d'applications continues: Si la n-sphère de rayon 2 est déformée dans R^{n+1} en la sphère de rayon $\frac{1}{2}$, une sphère déformée intermédiaire a $2n$ points, images des extrémités de n diamètres orthogonaux, sur la sphère de rayon 1.

Si f applique S^{n-1} dans R^l, l'hypothèse: k premier impair et $l<2\times$partie entière de $\left(\dfrac{n-1-\varepsilon(n)}{k-1}\right)$, où $\varepsilon(2m)=1$, $\varepsilon(2m+1)=0$, entraîne l'existence d'un k-uple de points orthogonaux de S^{n-1} ayant une image commune par f. La preuve de ce dernier théorème utilise l'action du groupe cyclique d'ordre k sur la variété de Stiefel des k-uples orthogonaux de S^{n-1}, ainsi que la théorie de Smith.
R. Deheuvels (New Haven, Conn.)

3862:

Smale, Stephen. The classification of immersions of spheres in Euclidean spaces. Ann. of Math. (2) **69** (1959), 327–344.

The author classifies the immersions of a k-sphere S^k into euclidean n-space E^n for arbitrary k and $n>k$ under the equivalence relation of "regular homotopy". Two immersions f, f': $S^k \to E^n$ (differentiable maps with Jacobian matrix of maximal rank) are said to be regularly homotopic if there exists a homotopy $h\colon S^k \times I \to E^n$ between f and f' such that $h|S^k \times (t)$ is an immersion for every t and the induced homotopy of the tangent bundle is continuous.

The main result is that the regular homotopy classes of immersions $f\colon S^k \to E^n$, $k<n$, are in one-to-one correspondence with the elements of the homotopy group $\pi_k(V_{n,k})$, where $V_{n,k}$ is the Stiefel manifold of k-frames in euclidean n-space. (The theorem is stated and proved for "based immersions" but is valid in the above form. This is mentioned by the author for $n>k+1$.)

This result is not only important for immersions of spheres but is also basic for the problem of classifying

immersions of arbitrary manifolds as seen in M. Hirsch, Trans. Amer. Math. Soc. **93** (1959), 242–276.

Some applications are given. One of them is the theorem: There exists an immersion of S^k into E^{k+1} with normal degree zero if and only if S^k is parallelizable. This solves a problem originally considered by H. Hopf and subsequently by J. Milnor [Comm. Math. Helv. **30** (1956), 275–284; MR **18**, 60].

The correspondence between regular homotopy classes of immersions $f: S^k \to E^n$ and $\pi_k(V_{n,k})$ is as follows: Let $s: S^k \to E^n$ be an immersion regularly homotopic to the standard imbedding $S^k \subset E^{k+1} \subset E^n$ and such that $s|U = f|U$, where U is some spherical neighborhood of a point $x_0 \in S^k$. Let F_k be a field of tangent k-frames on $S^k - x_0$. Define $\omega: S^k \to V_{n,k}$ by

$$\omega(x) = \begin{cases} df[F_k(rx)] & \text{if } x \in E_+{}^k, \\ ds[F_k(rx^*)] & \text{if } x \in E_-{}^k, \end{cases}$$

where $r: E_+{}^k \to S^k - \bar{U}$ is a diffeomorphism and $x \to x^*$ is the symmetry with respect to the equator $E_-{}^k \cap E_+{}^k$ of S^k ($E_+{}^k$, $E_-{}^k$ = northern and southern hemispheres of S^k). Then $\Omega(f)$ = the homotopy class of ω is the element in $\pi_k(V_{n,k})$ corresponding to the class of f.

The proof that Ω is injective and surjective is by showing that the spaces $\Gamma_{k,n}$ (= space of C^2-immersions of the k-disk D^k into E^n which agree on $\dot{D}^k$ up to first order derivatives with the standard imbedding, with the C^2 topology) and $\Gamma_{k,n}'$ (= space of mappings $D^k \to V_{n,k}$ which are given constant on the boundary with the CO-topology) have the same weak homotopy type.

A mapping $\Phi: \Gamma_{k,n} \to \Gamma_{k,n}'$ is easily defined. The proof that Φ is a weak homotopy equivalence is by a difficult induction argument on k which involves proving that certain maps between immersion spaces satisfy the covering homotopy property.

As a byproduct of the proof: For $n > k+1$ an immersion $f: S^k \to E^n$ can be extended to an immersion $D^{k+1} \to E^n$ if and only if it is regularly homotopic to the standard imbedding (i.e., if and only if $\Omega(f) = 0$).

M. A. Kervaire (New York, N.Y.)

3863:

Kervaire, Michel A. Sur le fibré normal à une sphère immergée dans un espace euclidien. Comment. Math. Helv. **33** (1959), 121–131.

The main theorem characterizes sphere-bundles over the sphere S_d which can be realized as the normal bundle of some immersion of S_d in Euclidean space E_{d+n} ($n \geq 1$). Define

$$\partial: \pi_d(V_{d+n,d}) \to \pi_{d-1}(\mathrm{SO}(n))$$

to be the boundary operator coming from the fibration $\mathrm{SO}(d+n)/\mathrm{SO}(n) = V_{d+n,d}$. Then the set of $\alpha \in \pi_{d-1}(\mathrm{SO}(n))$ which correspond to normal bundles induced by an immersion $f: S_d \to E_{d+n}$ is the image of $\partial: \pi_d(V_{d+n,d}) \to \pi_{d-1}(\mathrm{SO}(n))$. The main tool is the theorem of the reviewer [see above review] that the classification of immersions of S_d in E_{d+n} up to regular homotopy is given by $\pi_d(V_{d+n,d})$. In proving his main theorem the author shows that addition in $\pi_d(V_{d+n,d})$ corresponds to joining two immersions by a tube. The reviewer [ibid.] posed the question as to when a regular homotopy class of S_d in E_{d+n} contained an imbedding. The author gives many values of d and n in the range $d < 2n-1$ for which only one such class contains an imbedding. Another theorem proved is that an immersion $f: S_d \to E_{d+n}$ is regularly homotopic to an immersion $g: S_d \to E_{d+m} \subset E_{d+n}$ ($m \geq 1$) if and only if there is a field of $(n-m)$-normal frames to $f(S_d)$ in E_{d+n}.

S. Smale (Princeton, N.J.)

3864:

Kotzig, Anton. Über die im Gleichgewicht gerichteten endlichen Graphen. Časopis Pěst. Mat. **84** (1959), 31–45. (Slovak. Russian and German summaries)

Let G be an oriented graph without isolated vertices. The author calls G equilibrated if the boundary of the sum of all edges (taken with their prescribed orientation) is zero. Various theorems are proved concerning mainly the number of all equilibrated orientations of a given graph, of all equilibrated subgraphs of a given oriented graph, etc. For some numbers considered, explicit formulas are given.

M. Katětov (Prague)

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 3513, 3516, 3518, 3687, 3813.

3865:

Wilker, Peter. Über den Vektor. Elem. Math. **14** (1959), 27–37.

The author in this article gives an elementary introduction to differential geometry using the classical tensor approach.

L. Auslander (Bloomington, Ind.)

3866:

★Alexandrow, A. D. Kurven und Flächen. VEB Deutscher Verlag der Wissenschaften, Berlin, 1959. 82 pp. Brosch.: DM 4.00.

Translation, by B. Heise and H. Kuhnke, of Chap. 7, Vol. 2 of *Matematika: ee soderžanie, metody i značenie*, Izdat. Akad. Nauk SSSR, Moscow, 1956 [MR **19**, 520]; elementary, expository discussion.

3867:

Wunderlich, W. Äquidistante Paare ebener Kurven mit konstanter Schränkung. Monatsh. Math. **63** (1959), 271–276.

Two curves in euclidean 3-space form an equidistant pair if they have a family of common normals. The equidistant pairs of plane curves are studied which have the property that corresponding tangents form a fixed angle. They are the meridians of surfaces of revolution of constant positive curvature [cf. Wunderlich, Arch. Math. **10** (1959), 64–70; MR **21** #2249].

P. Scherk (Toronto, Ont.)

3868:

Künneth, Hermann. Dualisierbare Kurven im R^3. J. Reine Angew. Math. **201** (1959), 84–99.

A dualizable plane curve is a curve with a continuous tangent which is the envelope of its tangents. Rosenthal showed [Math. Ann. **78** (1913), 480–521] that such a

curve is the union of a countable number of convex arcs. One of the principal aims of the present paper is the generalization of this result to E^3. A dualizable space curve must satisfy two duality conditions: (1) The obvious one, that the curve has a continuous tangent and a continuous osculating plane; and that each tangent is the limit of intersections of osculating planes, each point is the limit of the intersections of the tangent at the point with osculating planes at neighboring points. (2) If T and σ are tangent and osculating plane at p, T' and σ' those at p', then the set of lines intersecting T and T' tends for $p' \to p$ to the set of lines through p or in σ. The plane through T' and p tends to σ and $T' \cap \sigma \to p$.

A dualizable space curve consists of a countable number of arcs of order 3; moreover, each tangent trace (the curve formed by the intersections of tangents of the curve with a plane in general position) is a plane dualizable curve.

A large part of the paper consists of the classification of the singularities of a dualizable curve with respect to its Frenet frame and their relations to the singularities of the tangent traces. A description of the results would require too many definitions.

H. Busemann (Los Angeles, Calif.)

3869:

Fässler, Walter. Über die Normaltorsion von Flächen im vierdimensionalen euklidischen Raum. Comment. Math. Helv. **33** (1959), 89–108.

Let S be a surface in the 4-dimensional euclidean space R^4. The normal curvature and normal torsion of S along a tangent direction are defined by means of the curves of intersection of S with its normal hyperplanes in a natural manner. The author first gives the formulas of these two invariants and then applies them to a special class of surfaces which are characterised by the property that any two tangent planes are equally inclined (two planes in R^4 are called equally inclined if the orthogonal projection from one to the other preserves the angle). For such a surface, it is proved that (i) the Gaussian curvature is equal to the negative of twice the square of the normal curvature, and (ii) the normal torsion can be expressed in terms of the Gaussian curvature and its first partial derivatives.

H. C. Wang (Evanston, Ill.)

3870:

Bartoševič, M. A. Plane-parallel webs of hypersurfaces. Dokl. Akad. Nauk SSSR **124** (1959), 970–972. (Russian)

Un p-réseau dans un espace à n dimensions ($n<p$) est l'ensemble de p familles à un paramètre d'hypersurfaces. Des p-réseaux, qui se correspondent par les transformations ponctuelles d'un groupe analytique continu, sont équivalents. Les p-réseaux équivalents à p familles d'hyperplans parallèles sont appelés réseaux plan-parallèles. Ces réseaux ont été étudiés par Blaschke et ses élèves [W. Blaschke, *Einführung in die Geometrie der Waben*, Birkhäuser, Basel-Stuttgart, 1955; MR **17**, 780].

Dans la présente note, l'auteur, utilisant les méthodes développées par Laptev et Vassilev, donne des caractères invariants des p-réseaux plan-parallèles dans un espace à n dimensions pour n et p arbitraires. Ces caractères se traduisent par la nullité de certains tenseurs relatifs ou de certains invariants relatifs dans l'espace de représentation du groupe considéré.

M. Decuyper (Lille)

3871:

Demaria, Davide Carlo. Su alcuni sistemi ∞^2 di rette in S_4. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **92** (1957/58), 457–470.

L'Autore dimostra che i sistemi di ∞^2 rette dello spazio S_n tali che le loro coordinate di retta sono soluzioni di una stessa equazione di Laplace sono congruenze W contenute in un S_3 subordinato di S_n, oppure sono costituiti di ∞^1 fasci di rette.

D. Gallarati (Genoa)

3872:

Finikov, S. P. Voss surfaces in E_4. Dokl. Akad. Nauk SSSR **120** (1958), 1214–1216. (Russian)

In three-dimensional Euclidean space E_3, a Voss surface is one which admits two families of geodesics which at each point have directions which are conjugate (relative to the second fundamental form). These surfaces were given another characterization by Guichard in terms of congruences [cf. Bianchi, *Vorlesungen über Differentialgeometrie*, Teubner, Leipzig, 1899].

In this paper Voss surfaces, in the sense of the first definition, in E_4 are studied by the method of moving frames. It is shown that they admit deformations which preserve the defining property, i.e. the geodesics nets remain conjugate, and share other properties with the Voss surfaces in E_3. However, the characterization of Guichard is no longer valid in E_4.

W. M. Boothby (St. Louis, Mo.)

3873:

Akivis, M. A. Projective-metric theory of pairs of T-complexes. Mat. Sb. N.S. **46** (88) (1958), 399–420. (Russian)

The general background of the discussion in this paper is the representation of straight lines in a projective three-space P_3 by points of a quadratic hypersurface Q_4 in P_5 (due to Plücker). Projective transformations in P_3 correspond, as shown by Rozenfel'd [same Sb. **22** (64) (1948), 457–492; MR **10**, 66], to metric properties of P_5 (with Q_4 as the absolute). The particular situation considered in the present paper deals with T pairs of complexes introduced by the author earlier [Dokl. Akad. Nauk SSSR **65** (1949), 429–432; MR **11**, 134]. He has shown then that to such a T pair corresponds a focal family T of rays. A correlation with respect to Q_4 now permits one to assign to this family a three-dimensional surface S, metric properties of which correspond to projective properties of the T pair. The first and two second quadratic forms of S together with a linear form ω play an important part in the discussion. An interesting special case, in which the pair is called (because of analogy with Ribaucourt surfaces) an R pair, is characterized by any one of eight properties, such as the orthogonality of the conjugate net on S, developability of principal ruled surfaces on T, or the statement that ω is an exact differential.

G. Y. Rainich (Notre Dame, Ind.)

3874:

Lerda, P. Attilio. Invarianti proiettivi di due elementi curvilinei spaziali di ordini due e quattro. Boll. Un. Mat. Ital. (3) **14** (1959), 28–36.

Consider in the real projective space of n dimensions

an element of curve given in the neighborhood of a point x_0 by an expansion

$$x^i = x_0{}^i + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d_k x^i}{dt^k} (dt)^k.$$

Then an equivalence class of such elements of curves whose expansions coincide up to $d_k x^i/dt^k$ will be called an E_k.

In Atti Accad. Naz. Lincei. Rend. (6) **22** (1935), 483–491, Bompiani considered two curve-elements E_3 and E_3 in general position in 3-space, the corresponding projective invariant, and its geometric interpretation.

The author considers the corresponding question for an E_3 and an E_4, and expresses the 3 invariants he finds as rational functions of cross-ratios of 6 points on 3 twisted cubics related to the different E_i's present.

A. Gutwirth (Berkeley, Calif.)

3875:

★**Cazenave, René. Formes inédites du rotationnel et du champ vectoriel en général.** Actes du colloque de calcul numérique, Périgueux, 1957, pp. 1–3. Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. 80, Paris, 1959. vii + 87 pp. 1800 francs.

It is shown that if **H** is a continuously differentiable vector field: (1) there exists scalar fields U and V such that curl $\mathbf{H} = \text{grad } U \times \text{grad } V$; (2) if **H** is not a gradient, there exist a scalar field W and a vector field **K**, such that $\mathbf{H} = W$ curl **K**. *T. N. E. Greville* (Kensington, Md.)

3876:

★**Cazenave, René. Fonctions génératrices de certaines transformations ponctuelles.** Actes du colloque de calcul numérique, Périgueux, 1957, pp. 5–13. Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. 80, Paris, 1959. vii + 87 pp. 1800 francs.

It is pointed out that any vector field can be regarded as defining a point transformation (each point of the field being transformed into the point whose coordinates are the components of the corresponding vector). If the vector field is the gradient of a scalar point function, the latter function is here defined as the generating function of the transformation. Generating functions are deduced for radial transformations (those depending only on the vector from a fixed point to the transformed point), rotations, projective transformations, and Lorentz transformations.

T. N. E. Greville (Kensington, Md.)

3877:

Moór, Arthur. Über Tensoren, die aus angegebenen geometrischen Objekten gebildet sind. Publ. Math. Debrecen **6** (1959), 15–25.

It is proved that the most general pure contravariant tensor with r indices $t^{i_1\cdots i_r}$, which is constructed of only a contravariant vector $\xi^i(x^1, \cdots, x^r)$, has the form $t^{i_1\cdots i_r} = c(x)\xi^{i_1}\cdots\xi^{i_r}$, where $c(x)$ is a scalar. And similarly the most general tensor with two indices f_{ik}, which depends on only a covariant symmetric or skew-symmetric tensor g_{ab} and is continuous with respect to its arguments, has the form $f_{ik} = c(x)g_{ik}$ [see for the case of the tensor g_{ik} of the general type, M. Ikeda and S. Abe, Tensor (N.S.) **7** (1957), 59–69; MR **20** #5071a]. It is shown also that the tensor of the type $F_i{}^j{}_k$ which depends on only the connection parameters $\Gamma_i{}^j{}_k$ can be only functions of skew-symmetric parts of $\Gamma_i{}^j{}_k$. At the last, there are examples for tensors $T_i{}^j{}_{kl}$ which are constructed of $\Gamma_i{}^j{}_k$ and $\partial_l \Gamma_i{}^j{}_k$. The characteristic functional equation of the tensor $T_i{}^j{}_{kl}$ is given, but its most general solutions are not yet determined. This kind of problem was discussed first by S. Gołąb and M. M. Kucharzewski [Ann. Polon. Math. **2** (1955), 250–253; MR **17**, 661] but is not as yet solved completely. *A. Kawaguchi* (Sapporo)

3878:

Géhéniau, J. Sur les tenseurs du groupe de Pauli. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 418–422.

"The tensors quadratic functions of the components of a spinor and its hermitian conjugate are classified by the Pauli group in isoscalars and isovectors. Some identities are given. Relations between our formalism and Gürsey's [Nuovo Cimento (10) **7** (1958), 411–415] are mentioned." (Author's abstract) *M. Loève* (Berkeley, Calif.)

3879:

Beaufays, O. Sur la représentation géométrique des grandeurs spinorielles. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 555–576.

Looking first at the results laid down in the theorems of this paper the reader is shocked by such impossible concepts as: eigenvalue of a null polarity, the set of fixed points of a null polarity etc. This is caused by the fact that the author, without saying so, considers the product of these correlations and the polar correlation with respect to the quadric $x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2 = 0$, thus reducing the correlations to collineations! Taking this into account the contents of the paper can be summarised as follows: Between the ∞^3 point pairs $\psi \equiv (x_1, x_2, x_3, x_4)$, $\gamma_5\psi \equiv (x_1, x_2, -x_3, -x_4)$ and the tangents to the quadric a unique correspondence can be established. The special correspondence given in the paper is deduced by means of Dirac-matrices and their relations. It corresponds to nearly trivial relations in the Kummer configuration based on the 16 points $\gamma_\mu\psi$, $\gamma_\mu\gamma_\nu\psi$ and the plückerian coordinates of the tangents to the quadric.

E. M. Bruins (Amsterdam)

3880:

Guy, Roland. Sur la dérivation covariante des spineurs. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 512–519.

Recent progress in differential geometry in the large makes possible a new study of the covariant differentiation of spinors. The definition suggested in the paper makes possible the synthesis of riemannian tensors and spinors. It permits one to relate the covariant derivative of spinors to the covariant derivative of all tensors which provide different representations of a Kleinian geometry of group G. Thus a canonical procedure for differentiation of spinors is achieved. *D. E. Spencer* (Storrs, Conn.)

3881:

Cattaneo Gasparini, Ida. Campi basici e trasformazioni affini. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **25** (1958), 33–42.

3882:

Švec, Alois. L'élément linéaire projectif d'une surface plongée dans l'espace à connexion projective. Czechoslovak Math. J. **8** (83) (1958), 285–291. (Russian summary)

Author's summary: "Dans ce travail j'étudie l'interprétation géométrique des courbes d'une surface plongée dans l'espace à trois dimensions et à connexion projective, cettes courbes généralisant les courbes de Darboux. On définit les éléments linéaires projectifs, la préservation de ces éléments étant la condition nécessaire et suffisante pour la déformation du second ordre de la surface, ou de sa dualisation."

P. O. Bell (Culver City, Calif.)

3883:

Haimovici, Adolf. Sur une certaine correspondance conforme entre deux courbes. Czechoslovak Math. J. **9** (84) (1959), 297–304. (Russian summary)

Given a family of circles (k) touching a curve C in 3-dimensional conformal space, it is proved that there is a corresponding family of circles $(\bar{k})$ touching a curve $\bar{C}$ such that the spheres containing a circle k are orthogonal to the spheres containing the corresponding circle $\bar{k}$.

A. G. Walker (Liverpool)

3884:

Lumiste, I. G. The geometric structure of a complex-analytical surface V_{2n} in space R_{2N}. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 259–262. (Russian)

A complex-analytical V_{2n} in euclidean R_{2N} is expressed in orthogonal coordinates by the equation $x^J+ix^{\bar{J}}=f^J(u^k+iv^k)$, $J=1, \cdots, N$, $\bar{J}=N+J$, $k=1, \cdots, n$, where the $f(w^k)$ are analytical functions of n complex variables $w^1, \cdots, w^n$. For $n=1$, $N=2$ we thus have the minimal surfaces studied by Kommerell and Eisenhart (with circular normal indicatrix), for $n=1$ and N arbitrary those of O. Borůvka [Publ. Fac. Sci. Univ. Masaryk no. **214** (1935), 1–25]. For general n and N, such a V_{2n} is characterized by the properties (1) that it can be considered as a translation surface formed by two complex-conjugate fully isotropic analytical X_n, $\bar{X}_n$ which (2) lie respectively in two plane generators I_N, $\bar{I}_N$ of the isotropic cone of R_{2N}, intersecting only at a point of the surface. The V_{2n} thus belongs to the class of minimal surfaces with two isotropic conjugate directions I_n, $I_{\bar{n}}$. When $N=n+1$ the condition (2) can be replaced by the condition that when a tangent direction is turned in the 2-direction determined by two complex conjugate directions tangent, respectively, to X_n, $\bar{X}_n$, then the end of the corresponding normal curvature vector describes a circle in the two-dimensional normal plane (the circular normal indicatrix). [Reference is made to a dissertation by M. Z. Osipova, Moskov. Gor. Ped. Inst. Potemkin., 1954].

D. J. Struik (Cambridge, Mass.)

3885:

Lumiste, U. G. On surfaces V_n with multidimensional isotropic conjugate directions in spaces R_N or S_N. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 702–705. (Russian)

A surface V_n in euclidean R_N or non-euclidean S_N is said to have a complete tangent system of isotropic conjugate directions if its tangent plane at an arbitrary point contains totally isotropic directions I^κ of dimensions p_κ $(\kappa=1, \cdots, k)$, $\sum p_\kappa=n$, in a set not lying in a plane of dimension $m<n$ and weakly conjugate in pairs [in the sense of V. V. Ryžkov, Uspehi. Mat. Nauk (N.S.) **11** (1956), no. 4 (70), 180–181]. This is therefore a generalization of the ordinary minimal surface $(n=2)$. Analytically such a surface V_n is defined in R_N at a point M by $dM=\omega^a e_a$ $(a, b=1, \cdots, n)$, e_{i_κ} lying in the I^κ, e_α in planes normal to V_n, in such a way that $\omega^\alpha=0$, $\omega_a{}^\alpha=\Lambda_{ab}{}^\alpha\omega^b$, $\Lambda_{ab}{}^\alpha=\Lambda_{ba}{}^\alpha$, and $\Lambda^\alpha_{i_\kappa j_\lambda}=0$ $(\lambda\neq\kappa)$. Among the vectors $e_{ab}=\Lambda_{ab}{}^\alpha e_\alpha$, which determine the n_1-dimensional first normal planes, only the vectors $e_{i_\kappa j_\kappa}$ are different from zero, and $n_1\leq\sum\frac{1}{2}p_\kappa(p_\kappa+1)$. The directions I^κ can be decomposed into the largest groups such that the directions in one group are in the smallest common generating plane J^ρ $(\rho=1, \cdots, r)$ of the isotropic cone. It is then shown that nonisotropic V_n of the described type with maximal n_1 only exist in noneuclidean space if $n=2$. In the euclidean case such V_n also exist for $n>2$ and are surfaces of translation of fully isotropic submanifolds enveloped by the J^ρ, which themselves are stratified in families of submanifolds enveloped by the I^κ.

D. J. Struik (Cambridge, Mass.)

3886:

Couty, Raymond. Vecteurs et tenseurs invariants sur un espace homogène. C. R. Acad. Sci. Paris **246** (1958), 2569–2571.

In the first part of this short note, the author proves the following theorems. Theorem 1: A compact (or homogeneous) Hermitian space whose torsion tensor has vanishing covariant derivative with respect to Riemannian connexion is Kählerian. Theorem 2: In a homogeneous properly Riemannian space G/H, there does not exist a harmonic [Killing] G-invariant vector satisfying $R_{ji}\xi^j\xi^i\geqq 0$ $[R_{ji}\xi^j\xi^i\leqq 0]$ and having non-vanishing covariant derivative. If the Ricci curvature is positive [negative] definite, there does not exist a harmonic [Killing] vector field other than the zero vector. This theorem can be generalized to the case of harmonic [Killing] tensor fields.

We consider the case in which the cohomology algebra with respect to the operator d of G-invariant n-forms is of dimension one. We denote by $\mathscr{H}_{(n)}$ the homogeneous space satisfying this condition. Theorem 3: In a space $\mathscr{H}_{(n)}$ a G-invariant 1-form defines an isometry if and only if it is the solution of the equation $\nabla^i\nabla_i\xi^h+R_i{}^h\xi^i=0$.

In the second part of the note the author discusses a conformal Killing tensor $\xi_{i_1i_2\cdots i_p}$, that is, a skew-symmetric tensor satisfying

$$(*)\qquad \nabla_j\xi_{ii_2\cdots i_p}+\nabla_i\xi_{ji_2\cdots i_p}-\frac{2}{n}g_{ji}\nabla_r\xi^r{}_{i_2\cdots i_p}=0.$$

But the so-called conformal Killing tensor should be necessarily a Killing tensor for $p\geqq 2$, because, contracting $(*)$ by g^{ji_2}, we have

$$-\nabla_r\xi^r{}_{ii_3\cdots i_p}-\frac{2}{n}\nabla_r\xi^r{}_{ii_3\cdots i_p}=0,$$

that is, $\nabla_r\xi^r{}_{ii_3\cdots i_p}=0$ and consequently $\xi_{i_1i_2\cdots i_p}$ satisfies

$$\nabla_j\xi_{ii_2\cdots i_p}+\nabla_i\xi_{ji_2\cdots i_p}=0.$$

K. Yano (Tokyo)

3887:

Yano, Kentaro. Harmonic and Killing vector fields in compact orientable Riemannian spaces with boundary. Ann. of Math. (2) **69** (1959), 588–597.

L'A. étend aux variétés à bord certains des résultats obtenus par lui et S. Bochner [*Curvature and Betti numbers*, Princeton Univ. Press, 1953; MR **15**, 989] dans le cas des variétés compactes: par exemple, la non existence de champs de vecteurs harmoniques [resp. de Killing] sur les variétés à courbure de Ricci définie positive [resp. négative] est établie moyennant certaines conditions frontières; et il donne des conditions limites qui, jointes aux systèmes différentiels du deuxième ordre connus, permettent d'affirmer que la solution est un champ harmonique [resp. de Killing]. *J. Lelong* (Paris)

3888:

Vrănceanu, G. Sur une classe d'espaces symétriques. Deutsch. Akad. Wiss. Berlin. Schr. Forschungsinst. Math. **1** (1957), 112–123.

This paper was an address of the author at the Riemann-Tagung in Berlin, 1954. It deals with Riemannian spaces defined by a Hermitian metric, which have been introduced by Fubini [Ist. Veneto Sci. Lett. Arti. Atti. Cl. Sci. Mat. Nat. **63** (1904), 501–513]. Cartan [Bull. Soc. Math. France **54** (1926), 214–264; **55** (1927), 114–134] showed that these spaces are symmetric, i.e., the covariant derivative of their curvature tensor is zero. They have been rediscovered by the present author as the Riemannian spaces $V_{2p}(\lambda)$ which, after the spaces of constant curvature, admit a group of motions with a maximum number of parameters [Acad. R. P. Romîne. Stud. Cerc. Mat. **2** (1951), 387–444; MR **16**, 623].

Here, it is shown that the group of motions of a $V_{2p}(\lambda)$ has a simply transitive group if the space is of negative curvature ($\lambda<0$). This leads to a canonical form for the metric of the space which is a generalization of the well-known Beltrami formula for Riemannian spaces with constant negative curvature. It is pointed out that in this case ($\lambda<0$) the $V_{2p}(\lambda)$ is simply connected and can be defined, as the Euclidean space, by just one neighbourhood. Similarly there is given, for the metric of closed $V_{2p}(\lambda)$ with $\lambda>0$, a canonical form which is analogous to Riemann's formula for spaces with constant positive curvature. These $V_{2p}(\lambda)$ can be regarded as certain anholonomic varieties on the hypersphere S_{2p+1} and can be covered by $p+1$ neighbourhoods. *R. Blum* (Saskatoon, Sask.)

3889:

Vrănceanu, G. Researches of Soviet mathematicians on non-euclidean geometries. Acad. R. P. Romîne An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **13** (1959), no. 2 (29), 57–67. (Romanian. Russian summary)

Report on some results obtained by V. F. Kagan [Trudy Sem. Vektor. Tenzor. Analizu **7** (1949), 187–204; MR **12**, 276], P. A. Širokov [Mat. Sb. (N.S.) **41** (**83**) (1957), 361–372; MR **20** #2755] and B. A. Rozenfel'd [ibid., 373–380; MR **20** #2756] concerning spaces with linear element

$$ds^2 = \frac{d\bar{z}^i dz^i}{K\theta} - \frac{|z^i d\bar{z}^i|^2}{K\theta^2},$$

$$\theta = \bar{z}^i z^i \pm 1,\ i = 1, 2, \cdots, n,$$

where z and $\bar{z}$ are conjugate and z is either a real or complex number, or a quaternion $a+ib+jc+kd$; in this case $|z|=\sqrt{(a^2+b^2+c^2+d^2)}$. *D. J. Struik* (Cambridge, Mass.)

3890:

★Eum, Sang-seup. On the Hermite-Kaehler space. Kyungpook University Theses Collection, Vol. 1, pp. 237–244. Taegu, 1956.

The author obtains conformal and projective curvature tensors of a Kaehler metric analogous to those associated with a Riemannian metric. He proves that, corresponding to a given Kaehler metric there is a conformal Kaehler metric with zero Ricci tensor. *T. J. Willmore* (Liverpool)

3891:

Klingenberg, Wilhelm. Eine Kennzeichnung der Riemannschen sowie der Hermiteschen Mannigfaltigkeiten. Math. Z. **70** (1958/59), 300–309.

Let M^n be an n-dimensional differentiable manifold and E the bundle of frames. E is a principal bundle with groups and fibre $GL(n, R)$. Denote by E_G any (principal) subbundle obtained by reduction of the group to $G\subset GL(n, R)$, i.e., by restricting to frames for which the transition functions are in G. One may ask whether there is a linear connection on E_G uniquely determined by the condition that its torsion vanish. This is true when $G=O(u)$, for example, in which case the connection is the Levi-Civita connection of the Riemannian metric determined by the reduction. H. Weyl [*Mathematische Analyse des Raumproblems*, Springer, Berlin, 1923] answered this question in a complicated fashion, and it was reformulated and simplified by E. Cartan [J. Math. Pures Appl. **2** (1923), 167–192]. They proved that only when G is a real form of $O(n, C)$ does there exist a unique connection without torsion in every E_G.

The present paper proves the corresponding theorem for the case of an almost-complex manifold M^{2n}. In this case consider E^*, the principal bundle of group $GL(n, C)$ which determines the almost complex structure and $G\subset GL(n, C)$. Then there exists in each bundle E_G^* (obtained from E^* by reduction of the structure group to G) exactly one linear connection for which the mixed term (i.e., of type (1, 1)) of the torsion vanishes, if and only if G is a real form of $GL(n, C)$. *W. M. Boothby* (St. Louis, Mo.)

3892:

Matsumoto, Makoto. Relative Riemannian geometry. I. On the affine connection. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **31** (1958), 65–82.

Let M and N be differentiable manifolds of class C^∞, both n-dimensional, and let (x^a) and (y^i) be coordinate systems in M and N respectively. A tensor of (x)-order (p, q) and of (y)-order (r, s) is defined by its components $T^{a_1\cdots a_p i_1\cdots i_r}_{b_1\cdots b_q j_1\cdots j_s}(x, y)$, having the transformation law of tensors with respect to each index under coordinate transformations $\bar{x}^a=\bar{x}^a(x)$ and $\bar{y}^i=\bar{y}^i(y)$.

An affine connexion in M with respect to an observing point $Q(y)$ in N is defined, with respect to the so-called natural frame $\{P, e_i\}$, by the equations $dP=dx^a e_a$ and $de_b=\omega_b{}^a e_a$, where $\omega_b{}^a=\Gamma_{bc}{}^a(x, y)dx^c+C_{bk}{}^a(x, y)dy^k$. The Γ's are called translation-components and C's rotation-components. An affine connexion in N with respect to an observing point $P(x)$ in M is defined in a similar way.

The author next defines what he calls g-mapping. The g-mapping is a mapping which carries a vector dx^a in the tangent space to M at $P(x)$ to a vector dy^i in the tangent space to N at $Q(y)$, according to the equations $dy^i = g_a{}^i(x, y)dx^a$, where $g_a{}^i$ is a tensor of (x)-order $(0, 1)$ and of (y)-order $(1, 0)$ and $|g_a{}^i(x, y)| \neq 0$.

The affine connexion in M gives a mapping φ_M between tangent spaces at (x) and $(x+dx)$ and that in N gives a similar mapping φ_N. The author assumes that $\varphi_M g^{-1} = g\varphi_N$ which gives the relations between $\Gamma_{bc}{}^a$, $C_{bk}{}^a$, $\Gamma_{jk}{}^i$, $C_{jc}{}^i$ and $g_a{}^i$. Then the author defines the torsion tensor and curvature tensor for the general case and proves various identities satisfied by these tensors.

The author then restricts his considerations to the case in which the displacement dy of an observing point $Q(y)$ in N is obtained from the dx of a point $P(x)$ in M by the g-mapping. The torsion and the curvature obtained in this case are called g-torsion and g-curvature, respectively.

In the last section the author defines the paths in the following way. The parallelism of a vector v^a in M is defined by $dv^a + \Gamma_{bc}{}^a v^b dx^c + C_{bk}{}^a v^b dy^k = 0$. If the displacement dy^i of the observing point $Q(y)$ is obtained from dx of the point $P(x)$ by a g-mapping, then the above equations reduce to $dv^a + \Lambda_{bc}{}^a v^b dx^c = 0$, where $\Lambda_{bc}{}^a = \Gamma_{bc}{}^a + C_{bk}{}^a g_c{}^k$.

The author defines the path in M with respect to the observing point $Q(y)$ by the differential equations

$$\frac{d^2x^a}{dt^2} + \Lambda_{bc}{}^a \frac{dx^b}{dt}\frac{dx^c}{dt} = 0$$

and discusses its properties. *K. Yano* (Tokyo)

3893:

Matsumoto, Makoto. Relative Riemannian geometry. II. On the metric connections. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **31** (1958), 95–110.

In this second paper [see review above], the author assumes that a positive definite quadratic differential form $ds^2 = g_{ij}(x, y)dx^i dx^j$ is associated with every point $P(x)$ of M, where the coefficients $g_{ij}(x, y)$ are functions of (x) and furthermore depend upon (y) of the observing point $Q(y)$ of N. He then discusses metric connections with or without torsion. In the last two sections, he determines the so-called normal metric connection, which is quite similar to that in a Finsler space considered by E. Cartan, and discusses the geodesics in M and N.

K. Yano (Hong Kong)

3894:

Wilker, Peter. Invariante Grundlegung des affinen Raumes. Math. Ann. **137** (1959), 107–124.

The author in this paper begins by reviewing the concepts of differentiable manifold and affine connection. He goes on to give certain well known theorems concerning the special case where the connection is flat; i.e., has curvature and torsion zero. He first gives certain local results and finally certain global results. The global results serve to characterize the affine plane by means of differential geometry and topology. Some of the main results of this paper are also contained in a work by the reviewer and L. Markus [Ann. of Math. **62** (1955), 139–151; MR **17**, 298]. *L. Auslander* (Bloomington, Ind.)

3895:

Otsuki, Tominosuke. Tangent bundles of order 2 and general connections. Math. J. Okayama Univ. **8** (1958), 143–179.

The author in this article is interested in trying to present the theory of connections in such a way that the connections will be defined by a cross-section in an appropriate bundle. In order to do this he must enlarge the tangent bundle (i.e., the first order approximation to the manifold) to form a new bundle which includes the first and second order approximation or, in the language of Ehresmann, includes the jets of the first and second order. In a bundle of this sort the connection can be defined by a cross-section. This then permits the author to discuss the relation of a connection to a tensor of type (1, 2). The author also uses his construction to study the derivations in the graded algebra of tensor fields. In particular, he shows how the trivial differential operator on this algebra may be arrived at as covariant differentiation. The exact details are much too complicated to be given in this review. *L. Auslander* (Bloomington, Ind.)

3896:

Okubo, Tanjiro. On the existence of the contact metric spaces allowing extended plane transformations. Tensor (N.S.) **8** (1958), 151–158.

This paper is a sequel to another [Tensor (N.S.) **6** (1956), 32–59; MR **18**, 231] on what the author calls contact metric spaces in which there appear two sets of contact frames. The vanishing of either set of contact frames is a property which is not preserved for a homogeneous contact transformation. The spaces for which the second set of contact frames vanish and for which the group of transformations is an extended point transformation have already been studied under the name of C-spaces [K. Yano and E. T. Davies, Ann. Mat. Pura Appl. (4) **37** (1954), 1–36; MR **16**, 626]. In this paper and its predecessor the author makes a corresponding study of spaces for which the first set of contact frames vanish and which he calls C^1-spaces.

E. T. Davies (Southampton)

3897:

Stojanović, Rastko. Some theorems on intransitive groups of motions. Acad. Serbe Sci. Publ. Inst. Math. **10** (1956), 97–100.

Starting from a theorem of L. Bianchi and a theorem of G. Fubini on intransitive groups of motions in Riemannian spaces V_n the author demonstrates the theorems: 1. If a Riemannian space V_n admits a family of geodesically parallel hypersurfaces V_{n-1} of constant curvature, then V_n admits an intransitive group G_r of motions, $r = \frac{1}{2}n(n-1)$, for which the hypersurfaces V_{n-1} are the minimum invariant varieties. 2. If a Riemannian space of constant curvature V_n admits an intransitive group of motions G_{r_k}, where $r_k = \frac{1}{2}(n-k+1)(n-k)$ is the order of the group, the minimum invariant varieties of the group G_{r_k} are subspaces V_{n-k} of constant curvature and geodesically parallel with respect to some enveloping subspaces V_{n-k+1} of V_n, which are also of constant curvature. 3. The group G_{r_k} from the theorem 2 is the group of stability of a $(k-1)$-dimensional totally geodesic subspace V_{k-1} of V_n.

V. Dumitraş (Zbl **72**, 170)

3898:

Tashiro, Yoshihiro. A theory of transformation groups on generalized spaces and its applications to Finsler and Cartan spaces. J. Math. Soc. Japan **11** (1959), 42–71.

In this paper, the author develops the theory of Lie derivatives of a linear connection over a general tensor bundle space, and thus gives a unified method to discuss the group of automorphisms of the generalized spaces (in the sense of differential geometry). As applications, all the Finsler spaces and Cartan spaces of dimension n which admit a group of motions of order $\geqq \frac{1}{2}n(n-1)+1$ are determined, with the geometrical objects explicitly given in terms of local coordinates.

H. C. Wang (Evanston, Ill.)

3899:

Tanaka, Noboru. Projective connections and projective transformations. Nagoya Math. J. **12** (1957), 1–24.

"The main purpose of the paper is to establish a theorem concerning the relation between the group of all projective transformations on an affinely connected manifold and the group of all affine transformations. We shall say that an affine connection satisfies condition (E), if it is without torsion and affinely complete and if the Ricci tensor field $S(X, Y)$ is parallel. Our theorem states that if an affine connection satisfies condition (E) and if the quadratic form $S(X, X)$ is zero or not negative semi-definite, then the two groups coincide. This is just a generalization of the case of ordinary affine space which is well known in analytic geometry. The proof is based on the theory of normal projective connection introduced by Elie Cartan; in particular, we make use of the 'developing' process of this connection." (From the author's summary)

C. Longo (Parma)

3900:

★**Busemann, Herbert. Convex surfaces.** Interscience Tracts in Pure and Applied Mathematics, no. 6. Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London; 1958. ix+196 pp. $6.00.

"The purpose of this tract is to acquaint the mathematical public with a subject, convex surfaces, which during the past 25 years has experienced a striking and beautiful development. The reader will find a self-contained description of the main results of the theory: all definitions are given and all theorems are formulated precisely. Chapters I and II form in many respects a natural complement to Bonnesen and Fenchel's *Theorie der konvexen Körper* [Springer, Berlin, 1934]. They deal largely with subjects conjectured or suggested in that report. Proofs found in the report are not repeated. This material appears here for the first time in book form. Most of the content of the last two chapters can be found in books, mainly in A. D. Alexandrov's *Die innere Geometrie der konvexen Flächen*" [Akademie-Verlag, Berlin, 1955; MR **17**, 74].

Table of contents: I. Extrinsic Geometry. 1. Notations, terminology, basic facts. 2. Convex curves. 3. The theorems of Meusnier and Euler. 4. Extrinsic Gauss curvature. 5. The influence of the curvature on the local shape of a surface. II. The Brunn-Minkowski Theory and its Applications. 6. Mixed volumes. 7. The general Brunn-Minkowski theorem. 8. Minkowski's problem. 9. Uniqueness for given $\{R_1, \cdots, R_m\}$. III. Intrinsic Geometry. 10. Intrinsic metrics. 11. The metrics of convex hypersurfaces. 12. Differentiability properties of geodesics. 13. Angles. The convexity condition. 14. Triangulations. Intrinsic curvature. 15. The Gauss-Bonnet theorem. Quasigeodesics. IV. Realization of Intrinsic Metrics. 16. The rigidity of convex polyhedra. 17. The realization of polyhedral metrics. 18. Weyl's problem. 19. Local realization of metrics with non-negative curvature. 20. Existence of open surfaces. The general gluing theorem. 21 Monotypy of closed surfaces. 22. Other monotypy theorems. Deformations. 23. Smoothness of realizations. V. Conclusion.

The first chapter deals with the direct extrinsic geometry of a convex hypersurface in euclidean n-space E^n. As a rule, no differentiability assumptions are made. But even without them, e.g., Euler's theorem is shown to be valid almost everywhere [Busemann-Feller]. In 7, Alexandrov's second proof of the Fenchel-Alexandrov inequalities is outlined. The treatment of Minkowski's problem in terms of set functions on the unit-hypersphere follows the paper by Fenchel and Jessen. In the first three sections of Chapter III the intrinsic metric of a metric space, in particular of a convex hypersurface in E^n, is introduced (Alexandrov called a metric intrinsic if any two points a, b of the space can be connected by rectifiable curves γ and if their distance ab is equal to the greatest lower bound of the lengths of these γ's).

From Section 13 (p. 94) on, only convex surfaces C in E^3 and two-dimensional manifolds M with intrinsic metrics are considered. The mathematical style also changes. The first part was fairly detailed. But from now on "... we merely report partly without proofs with the intention of providing the reader with a clear understanding of the guiding principles: the main results are very beautiful because they supply, without unnatural restrictions, easily grasped answers to deep problems. But the proofs which are at present available are among the longest in the mathematical literature."—C is said to realize M if there is a mapping of M onto C which preserves the intrinsic distances. M is called monotypic if there exists one and up to congruences only one realization of M in the class of the C's. Following Alexandrov those M's are characterized which can be realized through closed convex polyhedra. In Section 21 Pogorelov's famous result is outlined that every closed C is monotypic. In most of the Conclusion, recent results of Alexandrov and his school on M's of "bounded curvature" are discussed.

The author uses a considerable variety of tools: classical differential geometry, functions of real variables and set functions, elliptic partial differential equations, integral equations and quadratic forms.

The reviewer hopes that this book will stimulate geometrical research on this continent.

P. Scherk (Toronto, Ont.)

3901:

Baudoin-Gohier, Simone. Extension des équations de Codazzi et des conditions de rigidité à des surfaces convexes différentiables de classe C^n ($n<3$). C. R. Acad. Sci. Paris **248** (1959), 2704–2706.

The equations of Codazzi are valid in an integrated sense for surfaces of class C^2 [Hartman and Wintner, Amer. J. Math. **72** (1950), 757–774; MR **12** 357; p. 759]. The author remarks that this implies that the Minkowski integral relation and its generalization by Herglotz are valid for surfaces of class C^2. Hence, Herglotz's proof for the rigidity of ovaloids is valid for surfaces of class C^2.

(Similar remarks have been made by A. D. Aleksandrov [cf. Efimov, Uspehi Mat. Nauk (N.S.) **3** (1948), no. 2 (24), 47–158; Amer. Math. Soc. Transl. No. 37 (1951); see p. 60; MR **10**, 324] and Wintner [Amer. J. Math. **74** (1952), 198–214; MR **13**, 864; see p. 207].) The author notes that analogous comments hold for convex surfaces of class C^1 having a continuous mean curvature as defined by Wintner [ibid. **78** (1956), 117–136; MR **19**, 977; see p. 118] in terms of integral relations. *P. Hartman* (Baltimore, Md.)

3902:

Sun', Hè-Šen. On the rigidity of an unclosed surface of non-negative curvature in the case of a non-orthogonal bushing linkage. Dokl. Akad. Nauk SSSR **121** (1958), 229–232. (Russian)

The paper contains generalizations of results due to Vekua and to the author. The former considered the case of orthogonal linkage and the latter a more general linkage but only for surfaces of revolution. Using two lemmas based on consideration of infinitesimal bending the author derives conditions for rigidity of a truncated ovaloid for the case of one opening and the case of several circular openings whose axes meet in one point.
G. Y. Rainich (Notre Dame, Ind.)

3903a:

Pa, Chen-Kuo. On the integral curvature of a closed space curve. Acta Math. Sinica **6** (1956), 206–214. (Chinese. English summary)

3903b:

Pa, Chen-Kuo. On the integral curvature of a curvilinear polygon. Acta Math. Sinica **7** (1957), 277–284. (Chinese. English summary)

3903c:

Pa, Chen-kuo. On the integral curvature of a curvilinear polygon. Sci. Sinica **7** (1958), 11–18.

[The third paper is a translation of the second.]

The author proves several inequalities on the total curvature of closed space curves. Some of the methods are interesting but most of the results are closely related to the work of I. Fáry [Bull. Soc. Math. France **77** (1949), 128–138; MR **11**, 393] and W. Fenchel [Bull. Amer. Math. Soc. **57** (1951), 44–54; MR **12**, 634].
C. J. Titus (Ann Arbor, Mich.)

3904:

Prvanović, Mileva. Les transformations conformes et projectives d'espace riemannien généralisé au sens de T. Takasu. Univ. Beogradu. Godišnjak Filozof. Fak. Novom Sadu **3** (1958), 265–272. (Serbo-Croatian. French summary)

Der Verf. untersucht die konformen und projektiven Transformationen eines im Sinne von T. Takasu [Yokohama Math. J. **5** (1957), 115–169; MR **20** #6136] verallgemeinerten Riemannschen Raumes. Es handelt sich dabei um Räume deren metrische Form von der Gestalt

$$ds^3 = g_{ijk}dx^idx^jdx^k$$

ist. Einige Sätze werden bewiesen.
T. P. Anđelić (Belgrade)

3905:

Ōtsuki, Tominosuke. Note on curvature of Finsler manifolds. Math. J. Okayama Univ. **8** (1958), 107–116.

S. B. Myers [Duke Math. J. **8** (1941), 401–404; MR **3**, 18] proved that if M is a complete Riemann manifold with mean curvature everywhere greater than or equal to e^2 then the diameter of M is less than or equal to π/e. The reviewer gave a generalization of this theorem to Finsler manifolds [Trans. Amer. Math. Soc. **79** (1955), 378–388; MR **17**, 190]. The author in this paper re-examines the reviewer's proof and has succeeded in removing the following two objectionable features in the original proof: (1) Use of geodesic coordinate systems; (2) use of auxiliary spaces in the definition of mean curvature.
L. Auslander (Bloomington, Ind.)

3906:

Hashiguchi, Masao. On parallel displacements in Finsler spaces. J. Math. Soc. Japan **10** (1958), 365–379.

The present paper represents a comparative study of the various covariant derivatives which have been defined in Finsler spaces. The author begins with a discussion of a space of line elements in which a metric tensor g_{ij} and connection coefficients $\Gamma_h{}^i{}_k$, $C_h{}^i{}_k$, satisfying certain assumptions, are given. In particular, it is assumed that $C_j{}^i{}_k(y)y^j = C_j{}^i{}_k y^k = 0$. Conditions which ensure that the connection be metrical are derived, and it is shown by means of two lemmas how further connections—not necessarily metrical—may be derived. The metric tensor is then subjected to further conditions which imply that the metric is that of a Finsler space. The author's connection then becomes that of Cartan [*Les espaces de Finsler*, Gauthier-Villars, Paris, 1934], from which those of Synge [Trans. Amer. Math. Soc. **27** (1925), 61–67], Taylor [Trans. Amer. Math. Soc. **27** (1925), 246–264], and Barthel [Arch. Math. **4** (1953), 346–365; MR **15**, 556] are derived. The application of the above-mentioned lemmas also yields the (distinct) parallel displacements as defined by Berwald [Math. Z. **25** (1926), 40–73] and the reviewer [Math. Z. **54** (1951), 115–128; MR **13**, 159].
H. Rund (Durban)

3907:

Eliopoulos, H. A. Subspaces of a generalized metric space. Canad. J. Math. **11** (1959), 235–255.

H. Rund [same J. **8** (1956), 487–503; MR **18**, 333] developed the differential geometry of a hypersurface of $n-1$ dimensions imbedded in a Finsler space of n dimensions, considered as locally Minkowskian. As an extension of the above-stated results, on making use of the connection parameters of Rund's type, the theory of m-dimensional subspaces F_m imbedded in a Finsler space of n dimensions $(m<n)$ is developed. That is, normal curvatures of F_m are defined and covariant derivatives of the normal vectors are deduced. At the last, the Gauss and Codazzi equations for an F_m are found.
A. Kawaguchi (Sapporo)

3908:

Kashiwabara, Shobin. On Euclidean connections in a Finsler manifold. Tôhoku Math. J. (2) **10** (1958), 69–80.

In a new language, the author expresses and discusses the euclidean connections in a Finsler manifold defined by S. S. Chern [Proc. Nat. Acad. Sci. U.S.A. **29** (1943), 33–37; MR **4**, 259] and gives a geometrical interpretation to the connections. *A. Kawaguchi* (Sapporo)

PROBABILITY

See also 3340, 3776, 3944, 4069.

3909:

★Medgyessy, Pál; and Takács, Lajos. Valószínűségszámítás. [Theory of probability.] Technical Books in Mathematics. University Manual, C. v. Tankönyvkiadó, Budapest, 1957. 333 pp. 31.00 Ft.

The book consists of two independent parts. Part A (General probability theory) was written by P. Medgyessy; the author of part B (Stochastic processes) is L. Takács. The aim of the book is to supplement existing texts on probability theory by summarizing essential results and by illustrating their use by problems which are worked out in full detail. The problems are selected from a great variety of fields, for example physics, electrical engineering (in particular telephone engineering), astronomy, meteorology, biology. The usual problems on coin tossing and on casting dies are omitted; the selection of interesting problems from the physical sciences and from technical applications increases greatly the value of the book. It should be useful not only for engineering students (for whom it was primarily written) but also for students of mathematics and mathematical statistics who are interested in probability theory and its applications.

The ten chapters of part A (which contains nearly 300 problems) are entitled: (1) The object of probability theory. Events. Algebra of events. (2) The probability of events. (3) Random variables, distribution and density functions. (4) The most common discrete distributions. (5) The most common distribution functions. (6) Distribution functions of certain functions of random variables. (7) Typical measures for random variables. (8) The laws of large numbers. (9) Characteristic functions, generating functions. (10) Limit theorems.

Part B contains almost 100 problems and has three chapters: (11) Markov chains. (12) Markov processes. (13) Non Markov processes. This last chapter deals with recurrent processes, stationary processes, and secondary processes induced by a primary process.

E. Lukacs (Washington, D.C.)

3910:

Block, H. D.; and Marschak, Jacob. An identity in arithmetic. Bull. Amer. Math. Soc. **65** (1959), 123–124.

The authors report the following arithmetical identity: Let N denote the set of integers $\{1, 2, \cdots, n\}$; let M be a fixed subset of N and i a fixed element of M. For any permutation r of the set N let i_r denote that integer of N which is ranked in the ith position by the permutation r. Let $R(i, M)$ denote the set of those permutations which rank i before the other elements of M. Let n positive numbers $u_1, u_2, \cdots, u_n$ be given. Then

$$(*) \quad \left(\prod_{j=1}^{n} u_j\right) \sum_{r\in R(i,M)} \prod_{k=1}^{n} \left[\sum_{i=k}^{n} u_{i_r}\right]^{-1} = u_i\left[\sum_{j\in M} u_j\right]^{-1}.$$

It is stated that the identity becomes intuitively clear from certain probability considerations. However, a required theorem in measure theory could not be found in the literature and the authors resorted to an induction proof.

R. D. James (Vancouver, B.C.)

3911:

Geffroy, Jean. Contribution à la théorie des valeurs extrêmes. Publ. Inst. Statist. Univ. Paris 7 (1958), no. 3/4, 37–121.

This paper is the first part of a lengthy and detailed presentation of results announced earlier [C. R. Acad. Sci. Paris **244** (1957), 1712–1714; **245** (1957), 1215–1217, 1291–1293; **246** (1958), 224–226, 1154–1156; MR **19**, 466, 690; **20** #2817, #2818]. The remainder of the author's theory is to be given in a subsequent paper.

R. Pyke (New York, N.Y.)

3912:

Sankaranarayanan, G. A note on the equidistribution of sums of independent random variables. J. Indian Math. Soc. (N.S.) **22** (1958), 93–98.

Let $X_1, X_2, \cdots$ be independent random variables with common characteristic function $\varphi(t)$. The author, apparently unaware of the strong law of large numbers, proves that $n^{-1}\sum_1^n \exp(itX_j)$ converges to $\varphi(t)$ with probability one, with a similar oversight in the other theorems.

D. A. Darling (Ann Arbor, Mich.)

3913:

Chang, L. C.; and Fisz, M. Asymptotically independent linear functions of empirical distribution functions. Sci. Record (N.S.) **1** (1957), 335–340.

Let X_{ji}: $1\leqq j\leqq k$, $1\leqq i\leqq n_j$, be independent and identically distributed random variables from a continuous distribution function F. Let S_{jn_j} denote the empirical distribution function (d.f.) of the jth sample $(X_{j1}, X_{j2}, \cdots, X_{jn_j})$. Define $\eta_{ik}(x)=\sum_{j=1}^k b_{in_j}n_j^{1/2}S_{jn_j}(x)$ for $i=1, 2, \cdots, k-1$, where the b_{in_j} are real numbers, and set $A_{ik}^+=\max_x \eta_{ik}(x)$, $A_{ik}=\max_x |\eta_{ik}(x)|$. The authors prove that if (a) $\sum_{j=1}^k b_{in_j}n_j^{1/2}=0$, (b) $n_j/n_1\to a_j>0$ as $n_1\to+\infty$, and (c) $b_{in_j}\to B_{ij}$ as $n_j\to+\infty$, where $\sum_{j=1}^k B_{hj}B_{ij}=\delta_{hi}$, then

$$\lim_{n_1\to+\infty} P[A_{ik}^+ \leqq z_i: 1 \leqq i < k] = \prod_{i=1}^{k-1}(1-\exp(-2z_i^2)),$$

$$\lim_{n_1\to+\infty} P[A_{ik} \leqq z_i: 1 \leqq i < k] = \prod_{i=1}^{k-1}K(z_i),$$

where $K(z)=\sum_{j=-\infty}^{\infty}(-1)^j\exp(-2j^2z^2)$. By employing theorems of Prohorov [Teor. Veroyatnost. i Primenen. **1** (1956), 177–238; MR **18**, 943] the authors prove that the A_{ik}^+ $[A_{ik}]$ are asymptotically independent. The theorem then follows upon substitution of the known limiting distributions of A_{ik}^+ and A_{ik} [cf. Doob, Ann. Math. Statist. **20** (1949), 393–403; MR **11**, 43].

R. Pyke (New York, N.Y.)

3914:

Chang, L. C.; and Fisz, M. Exact distributions of the maximal values of some functions of empirical distribution functions. Sci. Record (N.S.) **1** (1957), 341–346.

The statistics A_{ik}^+ or A_{ik} of the above review provide possible distribution-free procedures for testing the hypothesis that all k samples have a common d.f. against certain alternatives. Consider the particular case of $b_{in_j}=(n_jn_{i+1}/N_iN_{i+1})^{1/2}$ for $1\leqq j\leqq i$, $b_{in_{i+1}}=-(N_i/N_{i+1})^{1/2}$ and $b_{in_j}=0$ for $i+2\leqq j\leqq k$, where $N_i=n_1+n_2+\cdots+n_i$. That is, consider, for example,

$$A_{ik}^+ = (N_in_{i+1}/N_{i+1})^{1/2}\max_x\left[N_i^{-1}\sum_{j=1}^{i} n_jS_{jn_j}(x)-S_{(i+1)n_{i+1}}(x)\right].$$

The asymptotic joint d.f. of the A_{tk}^{+} $[A_{tk}]$ is given in the paper reviewed above. In the present paper the authors obtain the exact joint distribution of the A_{tk}^{+} $[A_{tk}]$ for this special case. The method of proof is to show by means of combinatorial techniques that the A_{tk}^{+} $[A_{tk}]$ are mutually independent. An exact formula is then possible, since the marginal d.f. of A_{tk}^{+} $[A_{tk}]$ have been derived by Korolyuk [Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 81–96; MR **16**, 727]. *R. Pyke* (New York, N.Y.)

3915:

Vincze, István. Einige zweidimensionale Verteilungs- und Grenzverteilungssätze in der Theorie der geordneten Stichproben. Magyar Tud. Akad. Mat. Kutató Int. Közl. **2** (1957), 3/4, 183–209. (Hungarian and Russian summaries)

Let $F(x)$ and $G(x)$ be two distribution functions. Define the corresponding empirical dfs. $F_n(x)$ and $G_n(x)$ for independent samples of size n. Define further $M^+=\{x:\sup_{-\infty<y<\infty}(F_n(y)-G_n(y))=F_n(x)-G_n(x)\}$ and let $\xi_n=\inf_{x\in M}x$, where $M=M^+$ if $\sup_{-\infty<y<\infty}(F_n(y)-G_n(y))\neq 0$ and $M=M^+\cap\{x:\ x>\inf x_0:\ F_n(x_0)-G_n(x_0)<0\}$ otherwise. For the case $F=G$ the exact joint distribution of $(\sqrt{(n/2)}\sup_{-\infty<y<\infty}(F_n(y)-G_n(y))=k/\sqrt{(2n)},\ F_n(\xi_n+0)+G_n(\xi_n+0)=r/n)$ and of a similar random variable where $(F_n(y)-G_n(y))$ is replaced by its absolute value is derived for integers k, r. The limit distribution for $n\to\infty$, $k/\sqrt{(2n)}\to y>0$ and $r/2n\to z$, $0<z\leqq 1$ is also obtained. This paper is related to a paper of Z. W. Birnbaum and Ronald Pyke [Ann. Math. Statist. **29** (1958), 179–187; MR **20** #393]. Numerous misprints in the German text cause some difficulties in reading the paper. *L. Schmetterer* (Hamburg)

3916:

Bartoszyński, R. Some remarks on the convergence of stochastic processes. Studia Math. **17** (1958), 313–322.

Let $\xi(t)$, $0\leqq t\leqq 1$ be a process with independent increments and no fixed discontinuities. For each n, let $\xi_{n1},\cdots,\xi_{nk_n}$ be independent random variables, individually negligible for large n; let $t_{n0}=0<t_{n1}<\cdots<t_{nk_n}=1$ be a partition of [0, 1], of small mesh for large n; and define a process by setting $\xi_n(t)=\xi_{n1}+\cdots+\xi_{nk}$, where k is chosen so that $t_{n,k-1}<t\leqq t_{nk}$. The principal result is that the distribution of the process ξ_n converges to the distribution of ξ in the metric of Prohorov [Teor. Veroyatnost. i Primenen **1** (1956), 177–238; MR **18**, 943] if and only if the distribution of the random variable $\xi_n(t)$ converges to that of $\xi(t)$ in the metric of Lévy uniformly in t. The proof is based on Lemma 3.4 of Prohorov's paper. The author also investigates particular classes of partitions (t_{nk}) and extends a theorem of Skorohod [Dokl. Akad. Nauk. SSSR **104** (1955), 364–367; MR **17**, 1096]. *G. A. Hunt* (Ithaca, N.Y.)

3917:

Tsurumi, Shigeru. On the recurrence theorems in ergodic theory. Proc. Japan Acad. **34** (1958), 208–211.

Let $(X,\mathfrak{F},\mu)$ be a measure space with $\mu(X)<\infty$, and let T be a single-valued equi-measure transformation of X into X. It is proved (theorem 1) that, for every $E\in\mathfrak{F}$, the "recurrence time" $r(x)=r(x,E)$ (= the least positive integer r such that $T^rx\in E$) is an integrable function and $\lim_{n\to\infty}n^{-1}\sum_{j=0}^{n-1}r(T^jx)=1$ [or $\leqq 1$] for almost all $x\in E$ [or for almost all $x\in E$]. From this theorem the author deduces another proof of M. Kac's recurrence theorem [Bull. Amer. Math. Soc. **63** (1947), 1002–1010; MR **9**, 194]: $\mu(E)\leqq\int_E r(x)\mu(dx)\leqq\mu(X)$ for every $E\in\mathfrak{F}$, and T is ergodic if and only if $\int_E r(x)\mu dx)=\mu(X)$ for every $E\in\mathfrak{F}$ of positive measure. *K. Yosida* (Tokyo)

3918:

van Dantzig, D.; et Zoutendijk, G. Itérations markoviennes dans les ensembles abstraits. J. Math. Pures Appl. (9) **38** (1959), 183–200.

The authors study the iterations of a discrete-parameter stationary Markoff process with an abstract state space by generalizing the methods developed by van Dantzig to find absorption probabilities of such processes and using his generalized matrix calculus [Ann. Inst. H. Poincaré **14** (1955), 145–199; MR **17**, 867].

Let σ_E be a σ-field of subsets of an abstract space E. A stationary Markoff process with state space E is defined when a transition probability function P on $E\times\sigma_E$ is specified: for $x\in E$, $X\in\sigma_E$, $P_X{}^x$ is the probability of a transition from state x to a state in X in a single jump. Whether or not the process continues at any stage is decided, however, by chance, with a probability $z<1$ for continuation. Let $\{B_\lambda\}$ be a countable partition of E into σ_E-measurable subsets. An "iteration of length l in cell B_λ" occurs when for l consecutive moments the system remains in B_λ. An iteration is "complete" if it is not part of a longer iteration. There is a probability $1-s_{\lambda l}$ of the system meeting with a "catastrophe" on undergoing a complete iteration of length l in cell B_λ.

Now let $C_X{}^x$ be the probability of transition from state x to a state in X in any ($\geqq 0$) number of jumps and without catastrophe. It is shown that with $z<\frac{1}{2}$,

$$(*)\qquad C=(1-z)\sum_{n=0}^{\infty}\sum_{\substack{\lambda_1,\cdots,\lambda_n\\ \lambda_i\neq\lambda_{i+1}}}\prod_{i=1}^{n}\sum_{l=1}^{\infty}z^l s_{\lambda_i l}(I_{B_{\lambda_i}}P)^l.$$

In this $(I_B)_X{}^x$ is 1 or 0 according as x is or is not in $B\cap X$, the addition of functions M, N on $E\times\sigma_E$ is defined as usual, and their multiplication by

$$(MN)_X{}^x=\int_E M_{dy}{}^x N_X{}^y.$$

The similarity of the last to ordinary matrix multiplication justifies calling such functions "generalized matrices". It is shown that they form a normed linear algebra with unit. This fact is used in proving (*).

In (*) is stored information about the probabilities of different types of iterations. By way of examples, laws governing special types of iterations are derived from (*), as is also the main result in the earlier paper cited above. *P. Masani* (Providence, R.I.)

3919:

Krishna Iyer, P. V.; and Shakuntala, N. S. Cumulants of some distributions arising from a two-state Markoff chain. Proc. Cambridge Philos. Soc. **55** (1959), 273–276.

An approximate evaluation of the joint cumulants (up to fourth order) of the numbers of transitions 1 to 1 and 1 to 2 in a sample of n observations from a two-state Markov chain. *P. Whittle* (Cambridge, England)

3920:

Mihoc, Gheorghe. Fonctions d'estimation efficients pour les suites de variables dépendantes. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1 (49)** (1957), 449–456.

Consider a simple homogeneous Markov chain with transition probabilities $p_{ij}=P(x_n=a_j|x_{n-1}=a_i)$. Let n_j denote the number of occurrences of state a_j in the sample $x_1, x_2, \cdots, x_n$ and let n_{ij} denote the number of times that state a_j is preceded by state a_i in the same sample sequence. Romanovskiĭ [*Diskretnye Cepi Markova*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949; MR **11**, 445] has chosen to estimate the probabilities p_{ij} with the estimation function n_{ij}/n_i. The author demonstrates that estimator n_{ij}/n_i is asymptotically efficient in the classical sense of the theory of estimation [see Cramér, *Mathematical methods of statistics*, Princeton Univ. Press, 1946; MR **8**, 39]. *H. P. Edmundson* (Pacific Palisades, Calif.)

3921:

Ray, Daniel. Stable processes with an absorbing barrier. Trans. Amer. Math. Soc. **89** (1958), 16–24.

Suppose $\{X(t), t\geqq 0\}$ is a symmetric stable process on the real line with exponent α, $0<\alpha<2$, and initial value $X(0)=0$. It may be assumed that the paths $X(\cdot)$ are right continuous functions, bounded on finite t-intervals. Let I be an arbitrary open interval (a, b), finite or infinite, containing the origin. Denote by T the first passage time from I, that is,

$$T = \inf\{t \geqq 0|X(t) \notin I\}.$$

The place of first passage is the random variable $X(T)$. The transition probability densities of the α-absorbing barrier process on I are given by

$$P_0(x, t) = \frac{d}{dx}\Pr\{X(t) \leqq x;\ T > t\}.$$

In this paper some properties of P_0 are derived.

In the case $a=-\infty$ explicit formulas are given for the distribution of $X(T)$ and for $\int_0^\infty P_0(x, t)dt$. The methods used here do not yield explicit formulas for a and b finite.

In the general case it is shown that the resolvent is the sum of a singular part and a regular part,

$$\int_0^\infty e^{-\lambda t}P_0(x, t)dt = S_\lambda(|x|) - R_\lambda^+(x-a) - R_\lambda^-(b-x),$$

for $\lambda\geqq 0$ and $x\in(a, b)$. The singular part $S_\lambda(|x|)$ is the resolvent of the original process if $\lambda>0$, or its finite part if $\lambda=0$. It is shown that $S_\lambda(\xi)$, $R_\lambda^\pm(\xi)$ are completely monotonic for $\xi\in(0, \infty)$ if $\lambda>0$. If $\lambda=0$, then $-d/d\xi S_\lambda(\xi)$ and $-d/d\xi R_\lambda^\pm(\xi)$ are completely monotonic in $(0, \infty)$. In addition, the behavior of the resolvent is studied for x near the boundaries a and b.

The results are derived mainly from the Desiré-André equation which relates $P_0(x, t)$ to the transition probability densities of the original process. The explicit formulas in the case $a=-\infty$ use in addition a method of analytic continuation applied to

$$\Phi(z) = e^{-i\pi b}z^{-\alpha/2}E\{e^{-\lambda T}e^{izX(T)}\}.$$

J. Elliott (New York, N.Y.)

3922:

Sevast'yanov, B. A. Transient phenomena in branching stochastic processes. Teor. Veroyatnost. i Primenen. **4** (1959), 121–135. (Russian. English summary)

A particle has probability $p_n\Delta t+o(\Delta t)$ of being replaced by n particles in the time period Δt, $n=0, 2, 3, \cdots$. Put $p_1=-p_0-p_2-p_3-\cdots$, $f(x)=\sum_{n=0}^\infty p_nx^n$, $a=f'(1)$, $b=f''(1)$, $c=f'''(1)$. Let μ_t be the number of particles existing at time t. If the p_n are independent of t, then μ_t is a temporally homogeneous Markov process. From results of Yaglom [Dokl. Akad. Nauk SSSR **56** (1947), 795–798; MR **9**, 149] it follows that if $\mu_0=1$ and $a=0$, then $\lim_{t\to\infty} P\{\mu_t/M^*(t)<y\}=1-e^{-y}$, $y>0$, where $M^*(t)=E(\mu_t|\mu_t>0)$. The author points out that this result by itself has little practical value since in practice we cannot know that a is precisely 0. In order to provide a more meaningful result, he considers limit theorems for $a\to0$ and $t\to\infty$. The main result in this direction is Theorem 2: $\mathrm{Max}_{y\geqq0}|P\{(\mu_t/M^*(t))<y|\mu_t>0\}-(1-e^{-y})|\to0$ as $t\to\infty$, $a\to0$, uniformly for all processes such that $f''(1)\geqq b_0$, $f'''(1)\leqq c_0$, where b_0 is any fixed positive constant and c_0 is any fixed non-negative constant. The author also obtains a limiting distribution for $a\to0$, $t\to\infty$ and $\mu_0\to\infty$, where μ_0 is the initial number of particles. The reviewer notes that in this case the limiting distribution is, except perhaps for adjustment of parameters, identical with that found by W. Feller [Acta Bioth. Ser. A **5** (1939), 11–40; MR **1**, 22] for certain diffusion processes in connection with the "struggle for existence".

T. E. Harris (Santa Monica, Calif.)

3923:

Ibragimov, I. A. Some limit theorems for stochastic processes stationary in the strict sense. Dokl. Akad. Nauk SSSR **125** (1959), 711–714. (Russian)

Let $\{x_j, -\infty<j<\infty\}$ be a strictly stationary sequence of random variables, and define $\bar{S}_n=\sum_0^{n-1}x_j$. The sequence is called '(1) regular' if

$$(1)\qquad \sup_{A,B}|P\{AB\}-P\{A\}P\{B\}| \leqq \phi(k)P\{A\},$$

where A is a set depending on $\cdots, x_{j-3}, x_{j-2}$, B is one depending on $x_{k+j+1}, x_{k+j+2}, \cdots$, and $\phi(k)\to0$ with $1/k$. The sequence is '(2) regular' if (1) is satisfied without the second factor on the right. A central limit theorem has been proved by Rosenblatt [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 43–47; MR **17**, 635] under (2) regularity but without assuming stationarity. A sequence of m-dependent random variables is (1) dependent. Diananda [Proc. Cambridge Philos. Soc. **51** (1955), 92–95; MR **16**, 724] has proved central limit theorems for (not necessarily stationary) sequences of m-dependent random variables, generalizing results of earlier writers. The following are the principal results announced. (In the continuous parameter case, sums are to be replaced by integrals.) Under (2) regularity, every limit law of any centered normalized sequence $\{(\bar{S}_n/B_n)-A_n\}$ is stable. (Throughout the following, $E\{x_n\}=0$, $E\{X_n^2\}=1$.) Under (1) regularity, if $E\{|x_0|^{2+\delta}\}<\infty$ for some $\delta>0$, and if $D_n^2=\mathrm{Var}\ \bar{S}_n\to\infty$, then $\bar{S}_n/D_n$ is asymptotically normal. Under (1) regularity, and if $\sum\phi(k)^{1/2}<\infty$, then $\sigma^2=E\{x_0^2\}+2\sum_1^\infty E\{x_0x_j\}$ is well-defined, and, if $\sigma\neq0$, $\bar{S}_n/(\sigma_n^{1/2})$ is asymptotically normal. If the x_j's are mutually independent, with a common distribution, if $y_j=\sum_{-\infty}^\infty c_{k-j}x_k$, with $\sum c_k^2<\infty$, and if $\sum_0^{n-1}y_j$ has variance $B_n^2\to\infty$, then $\sum_0^{n-1}y_j/B_n$ is asymptotically normal. *J. L. Doob* (Urbana, Ill.)

3924:

Langebartel, R. G. Average initial velocity of the terminus of a two-dimensional linkage. Proc. Amer. Math. Soc. **10** (1959), 128–132.

Kluyver (Nederl. Akad. Wetensch. Proc. Ser. A 8 (1906), 341–350] has treated the random walk problem, making use of a definite integral involving the product of two Bessel functions. The author gives an extension of the method for this problem: given n particles A_i in a plane, successively connected by rigid freely hinged rods, A_0 being fixed; determine the mean initial velocity of A_n for any distribution of the initial pivotal angular velocities, provided A_0A_n has a given value. *O. Bottema* (Delft)

STATISTICS

See also 3911, 3915, 4051, 4055.

3925:

★Diamond, Solomon. Information and error: An introduction to statistical analysis. Basic Books, Inc., New York, 1959. xii + 307 pp. $5.00.

A poetic, non-mathematical presentation of classical statistical methods for psychologists. "Information", not in the sense of Shannon, but as "the perceived effect of the experimental variable" (p. 7).

I. R. Savage (Minneapolis, Minn.)

3926:

Rao, C. Radhakrishna. Maximum likelihood estimation for the multinomial distribution. Sankhyā **18** (1957), 139–148.

Let $\pi = (\pi_1, \pi_2, \cdots, \pi_k)$ be the probabilities of k mutually exclusive and exhaustive events, $A = \{\pi\}$ be an arbitrary but fixed set of admissible values of π excluding any π for which some $\pi_i = 0$. This paper gives proof of the strong consistency of the maximum likelihood estimate (m.l.e.) of π. Under the further assumption that $A = \pi(\theta)$ is a one-parameter family satisfying the condition that $I(\theta_0, \theta) = -\sum \pi_i(\theta_0) \log [\pi_i(\theta)/\pi_i(\theta_0)]$ $(I \geqq 0)$ is bounded away from zero whenever θ is bounded away from θ_0 (the true value), the strong consistency of the m.l.e. of θ is proven. Properties of the m.l.e. of θ are discussed.

F. C. Andrews (Athens, Ga.)

3927:

Kitagawa, Tosio. Successive processes of statistical controls. II. Mem. Fac. Sci. Kyushu Univ. Ser. A **13** (1959), 1–16.

[For part I, see same Mem. **7** (1952), 13–28; MR **14**, 997.] The author studies the characteristic function of the distribution of the output of a discrete time process of the following type. For a given setting of the control at the ith stage, the output is independent of the past. The mean of the process differs from the control setting by an unknown constant. At each stage, the control is regulated according to a linear function of the preceding observations. Two Robbins-Monro convergence results are given for a similar process control problem which differs from the above in that the mean of the process is an unknown function of the control setting. Heuristic extensions of these ideas are applied to the case of continuous controls.

H. Chernoff (Stanford, Calif.)

3928:

Chanda, K. C. On some simplification in the construction of similar regions. Calcutta Statist. Assoc. Bull. **8** (1959), 159–161.

3929:

Hanania, Mary I. A generalization of the Bush-Mosteller model with some significance tests. Psychometrika **24** (1959), 53–68.

Consider a two-alternative learning situation with r subjects and m trials for each subject. Random variables X_{ij} $(i = 1, 2, \cdots, r; j = 1, 2, \cdots, m)$ are defined to be 1 for success and 0 for failure. The author makes assumptions which lead to the transition probabilities for failure (the same for each subject)

$$q_{ij} = \theta^{\sum_{a=1}^{j-1} x_{ia}} u_j, \quad 0 < \theta \leqq 1, \quad 0 < u_j \leqq 1.$$

$\theta < 1$ describes the situation in which success is rewarded, with a consequent lowering of q_{ij}, while $\theta = 1$ describes the non-reward situation. The u_j are parameters which reflect factors other than the presence or absence of reward. This scheme reduces to that of Bush and Mosteller [Ann. Math. Statist. **24** (1953), 559–585; MR **15**, 449] if $u_j = b^{j-1}q$, where $0 < b < 1$, and q is the initial probability for failure. The theory of asymptotically similar tests of Neyman [Trabajos Estadistica **5** (1954), 161–168; MR **16**, 729] is adapted to the problem at hand in order to test $\mathrm{H}: \theta = 1$, Alt: $\theta < 1$. An example is given.

R. F. Tate (Seattle, Wash.)

3930:

Schmetterer, Leopold. Über nichtparametrische Methoden in der Mathematischen Statistik. Jber. Deutsch Math. Verein. **61** (1958), Abt. 1, 104–126.

The paper is based on a report dealing with recent developments in the theory of mathematical statistics with particular emphasis on unbiased estimation and non-parametric tests. The method of presentation is purely mathematical. While presumably intended to give the non-statistical mathematician an idea of some of the problems and results of mathematical statistics, the paper is also of interest to the mathematical statistician by providing a unified approach to the topics under discussion. *G. E. Noether* (Boston, Mass.)

3931:

Bhat, B. R. On the routine method of analysis of factorial designs. J. Karnatak Univ. **2** (1956), no. 1, 108–111.

A method for the manual calculation of the main effects and interactions in a factorial design is described.

L. Weiss (Ithaca, N.Y.)

3932:

Lafon, Monique. Coefficient d'efficacité d'un bloc incomplet partiellement équilibré. C. R. Acad. Sci. Paris **248** (1959), 3114–3115.

Oscar Kempthorne gave an expression for the efficiency factor of an incomplete block design with two associate classes. The author shows that keeping v, r, and k and $\lambda_1 + \lambda_2$ fixed this expression is maximized if $\lambda_1 = \lambda_2$. Therefore if a balanced incomplete block design with the

given parameters exists its efficiency is larger than that of any partially balanced block design with the given parameters v, r, k, $\lambda_1+\lambda_2$.

H. B. Mann (Columbus, Ohio)

3933:

Tikkiwal, B. D. An application of the theory of multiphase sampling on successive occasions to surveys of live-stock marketing. J. Karnatak Univ. **1** (1956), no. 1, 120–130.

This paper contains a numerical example of an estimation procedure developed by the author. The theory behind the development is contained in a paper to be published in a forthcoming issue of Ann. Math. Statist.

L. Weiss (Ithaca, N.Y.)

NUMERICAL METHODS

See also 3600, 3640, 3641, 3688, 3689, 3755, 3786, 3947, 3948, 3990.

3934:

Gaier, Dieter. Untersuchungen zur Durchführung der konformen Abbildung mehrfach zusammenhängender Gebiete. Arch. Rational Mech. Anal. **3** (1959), 149–178.

For conformal mapping of a given multiply-connected domain onto a canonical domain bounded merely by whole circles, Koebe proposed two methods of successive approximation, i.e., the iterating method and the iteration method, in which each step consists of a conformal mapping of a simply-connected domain. A method analogous to the former was introduced by the reviewer in the doubly-connected case and has been investigated in some detail by the author [J. Math. Mech. **6** (1957), 865–883; MR **20** #105]. In the present paper, the author notices that the basic ideal in these methods may be observed from the viewpoint of "raising up the inversibility of successive image domains". Thus, quantitative estimates are obtained for the errors of approximating functions in Koebe's methods. The results are then specialized to the doubly-connected case. Similar results are also derived by remembering especially that the convergence velocity of the reviewer's iterative method can be greatly accelerated by the insertion of successive inversions. On the other hand, with respect to conformal mappings of nearly annular or nearly circular ring domains onto annuli, deviations from the identity mapping are estimated.

Y. Komatu (Tokyo)

3935:

Belaga, È. G. Some problems involved in the calculation of polynomials. Dokl. Akad. Nauk SSSR **123** (1958), 775–777. (Russian)

The problem of efficient methods of computing polynomials has been studied by A. M. Ostrowski [*Studies presented to R. von Mises*, pp. 40–48, Academic Press, New York, 1954; MR **16**, 523] and T. S. Motzkin [Bull. Amer. Math. Soc. **61** (1955), 163 (abstract)]. The Newton-Horner scheme evaluates any $f(x)=a_0x^n+\cdots+a_n$ at the expense of n multiplications and n additions. The following results are obtained: for any n it is possible, by a specific algorithm, to obtain $f(x)$ at the expense of $[\frac{1}{2}(n+1)]+1$ multiplications and $n+1$ additions and it is not possible to obtain $f(x)$ using fewer than $[\frac{1}{2}(n+1)]$ multiplications and divisions and fewer than n additions and subtractions. The algorithm in question and the results obtained are similar to those of Motzkin.

John Todd (Pasadena, Calif.)

3936:

Lotkin, Mark. Note on the method of contractants. Amer. Math. Monthly **66** (1959), 476–479.

The method of contractants evaluates the determinant of a matrix $A=(a_{ij})$, $i, j=1, \cdots, n$, by the calculation of a sequence of matrices of successively lower order. $A_{k+1}\equiv(a_{ij}^{(k+1)})$ is obtained from the $A_k\equiv(a_{ij}^{(k)})$ $(i, j=1, \cdots, n-k;\ k=0, 1, \cdots, n-1)$ by the algorithm

$$a_{ij}^{(k+1)} = 1/a_{i+1,j+1}^{(k-1)}[a_{ij}^{(k)}a_{i+1,j+1}^{(k)}-a_{i,j+1}^{(k)}a_{i+1,j}^{(k)}]$$

for $i, j=1, \cdots, n-k-1$; with $a_{ij}^{(-1)}=1$, $a_{ij}^{(0)}=a_{ij}$. Then $A_{n-1}=a_{11}^{(n-1)}=\det A$. The method is ascribed to R. H. Macmillan [J. Roy. Aero. Soc. **59** (1955), 772–773]. This algorithm clearly breaks down if a zero element a_{ij}^k occurs anywhere in A_k except in the outside rows or columns. A method is given for removing internal zeros in A by forming a new matrix $A^{(i)}$ whose rows are suitable linear combinations of the rows of A, so that $\det A^{(i)}=\det A$ and $A^{(i)}$ has no interior zeros. If one of the contractants A_k developed from $A^{(i)}$ has an interior zero one must suitably modify $A^{(i)}$ and start again on the successive contraction. A numerical example is given.

A. M. Duguid (Providence, R.I.)

3937:

Bodewig, E. Die Inversion geodätischer Matrizen. Bull. Géodésique (N.S.) **1956**, 9–62.

Bei der Ausgleichung geodätischer Triangulationsnetze nach der Methode der kleinsten Quadrate entstehen lineare Gleichungssysteme, die folgende spezielle Bauart aufweisen. Sobald die Disposition des Triangulationsnetzes bekannt ist und bevor die geodätischen Messungen ausgeführt werden, ergeben sich Bedingungsgleichungen, die davon herrühren, dass die Winkelsummen in den geschlossenen Polygonen des Netzes und um die einzelnen Knotenpunkte herum ganzzahlige Vielfache von π sein müssen. Die zugehörige "Winkelmatrix" hat ganzzahlige Elemente und auch spezielle Struktur. Oft sind nur einzelne Diagonalen parallel zur Hauptdiagonalen besetzt. Boltz schlug vor, diese Winkelmatrizen zu invertieren, bevor die übrigen Gleichungen des Ausgleichsproblems in Angriff genommen werden, die von den trigonometrischen Identitäten im Netz herrühren und erst nach Ausführung der geodätischen Messungen aufgestellt werden können.

Dabei wird man wünschen, dass die Winkelmatrizen exakt invertiert werden, das heisst, dass die Elemente der Inversen rationale Zahlen sind. Friedrich und Jenne haben diesem Problem mehrere Arbeiten gewidmet; sie verwenden Graphen als topologisches Hilfsmittel zur Beschreibung der Struktur der Winkelmatrizen und benutzen Kettenbruchentwicklungen zur Berechnung der Inversen. Die speziellen und von den normalen Methoden der linearen Algebra abliegenden Methoden dieser Autoren haben sich aber nur schwer eingebürgert.

Der Verfasser bearbeitet nun die Aufgabe der Inversion der Winkelmatrix mit der Matrizenrechnung. Der Kern seiner Methode besteht darin, dass er zunächst den einfachsten Typ explizit invertiert, bei welchem nur 3

Diagonalen besetzt sind. Es werden einfache Rekursionsformeln aufgestellt und die Resultate in Tabellen niedergelegt. Komplizierter gebaute Winkelmatrizen invertiert er durch Ränderung der genannten einfachen Matrizen mit Hilfe einer Relation von Frobenius und Schur.

Es entsteht ein übersichtlicher Algorithmus, der ein Minimum an Rechenaufwand fordert. Der Hauptteil der Arbeit ist der Entwicklung und Erprobung dieses Algorithmus gewidmet. Mehrere und bis in alle Einzelheiten ausgearbeitete Beispiele beweisen die Wirksamkeit des Verfahrens auch für kompliziertere Triangulationsnetze.

Am Schluss der Arbeit werden Modifikationen dieses Rechenverfahrens diskutiert. Zuerst wird die Formel von Frobenius und Schur durch eine Formel von Sherman und Bartlett ersetzt, die sich besser zur Erfüllung gewisser spezieller Wünsche eignet. Sodann wird die Inversion von beliebigen ganzzahligen Matrizen (die nicht notwendigerweise die spezielle Bauart der Winkelmatrizen aufweisen) gestreift und der Algorithmus von Smith vorgeschlagen. Er ist im wesentlichen die aus der Elementarteilertheorie geläufige Methode, eine Matrix durch elementare Transformationen und durch Rechnen in einem algebraischen Ring auf Diagonalgestalt zu bringen. Endlich wird eine iterative Methode geschildert, welche an Stelle der Formel von Frobenius und Schur treten kann und auch die explizite Inversion der einfachen Winkelmatrizen benutzt. Diese Methode eignet sich vielleicht für das automatische Rechnen.

Die Arbeit stellt einen wichtigen Beitrag zu einem Gebiet der linearen Algebra dar, das bisher von den Mathematikern kaum bearbeitet wurde.

E. Stiefel (Zürich)

3938:

Pestel, E.; und Mahrenholtz, O. Zum numerischen Problem der Eigenwertbestimmung mit Übertragungsmatrizen. Ing.-Arch. **28** (1959), 255–262.

This paper presents a computational technique for finding the eigen-frequencies of a linear vibrating system. The given system is approximated by another system whose mechanical properties can be described by a $2r\times 2r$ matrix U and whose eigen-frequencies are the zeros of a r-dimensional sub-determinant $D(\omega)$ of U. A procedure is discussed for the evaluation of a quantity $R(\omega)$ proportional to $D(\omega)$ for prescribed values of ω. By graphical or numerical interpolation the zeros of $R(\omega)$ can then be found.

W. C. Rheinboldt (Syracuse, N.Y.)

3939:

Abian, Smbat; and Brown, Arthur B. On the solution of an implicit equation. Illinois J. Math. **3** (1959), 374–380.

This paper describes a method of constructing the solution $y=Y(x_1,\cdots,x_n)$ of an equation $g(x_1,\cdots,x_n;y)=0$ by successive approximations. The proofs presented are actually applications of the contracting mapping theorem though this is nowhere indicated. In line with this the main assumptions are that g is continuous in a precisely defined domain N of the $n+1$-dimensional space of its variables and that in N

$$0<C\leq\frac{g(x_1,\cdots,x_n;u)-g(x_1,\cdots,x_n;v)}{u-v}\leq D.$$

The error estimates associated with the contracting mapping theorem are proven in several forms and finally some generalizations and special cases are discussed.

W. C. Rheinboldt (Syracuse, N.Y.)

3940:

Schwerdtfeger, Hans. Notes on numerical analysis. I. Polynomial iteration. Canad. Math. Bull. **2** (1959), 97–110.

If $f(x)=0$ has only simple roots, polynomials $h(x)$ and $h_1(x)$ satisfying $h_1f-hf'=1$, identically, can always be found, e.g., by the method of undetermined coefficients. Moreover, for any polynomial $p(x)$, $H=h+pf$ and $H_1=h_1+pf'$, replacing h and h_1, satisfy the same identity. If $\phi=x+fh$, and α is a zero of f, then $0=\phi(\alpha)-\alpha=\phi'(\alpha)$, and for x_0 sufficiently close to α_1 the sequence defined by $x_{i+1}=\phi(x_i)$ has second-order convergence to α. With $p(x)$ suitably chosen,

$$\Phi=x+fH=x+hf+pf^2$$

defines a sequence with third-order convergence.

This technique is applied to the determination of iterating polynomials for reciprocation and for root extraction, and conditions of convergence are discussed.

A. S. Householder (Oak Ridge, Tenn.)

3941:

Brun, Viggo. An application of a "carpenter's curve" to Simpson formulas. Nordisk Mat. Tidskr. **7** (1959), 20–24.

This note discusses a geometrical interpretation of numerical integration formulas by successive polygonal approximations to areas. Mention is made of a similar geometrical process applied to cubes and tetrahedra but this is not used to establish integration formulas.

P. C. Hammer (Madison, Wis.)

3942:

Selmer, Ernst S. A note on the preceding paper by V. Brun. Nordisk Mat. Tidskr. **7** (1959), 25–26.

The author shows that methods established geometrically in the above paper may be derived analytically.

P. C. Hammer (Madison, Wis.)

3943:

Skvorcov, V. S. Error estimate of the solution of a system of partial differential equations by the method of finite differences. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 155–160. (Russian)

Using the concept of a fundamental matrix the author gives an error estimate for the solution by the finite-difference method of boundary value problems for an elliptical system of linear partial differential equations of second order with constant coefficients and homogeneous with respect to the order of differentiation in the case of two and three variables.

The author shows that for interior points of the lattice this error will be of the form $O(\sqrt{h})$ and for boundary points of the lattice in the mean $O\{(r+h)^2\}$ where r is an average thickness of the boundary layer; h is the interval of the lattice.

A. H. Stroud (Madison, Wis.)

3944:

★Черницкий, П. И. Таблицы вероятностей. [Černickiĭ, P. I. Probability tables.] Voennoe Izdat., Minist. Oboron. Soyuza SSR, Moscow, 1957. 223 pp. 5.45 rubles.

These are tables giving the least number of trials $N=f(p, P_i)$ necessary in order to ensure that the probability of having at least i successes exceeds P_i, the probability of success in a single trial being p. The tables are given for $i=1(1)15$, for $p=0.2(.2)10(1)98\%$ and for $P_i=0.05(.05)0.95$, 0.98. *John Todd* (Pasadena, Calif.)

COMPUTING MACHINES

3945:

★Китов, А. И.; и Криницкий, Н. А. Электронные вычислительные машины. [Kitov, A. I.; and Krinickiĭ, N. A. Electronic calculating machines.] Naučno-Populyarnaya Seriya. Izdat. Akad. Nauk SSSR, Moscow, 1958. 131 pp. 1.90 rubles.

This book is published by the Academy of Sciences, USSR, under the editorship of Academician A. A. Dorodnitsyn, with a printing of 25,000 copies. It is a satisfactory and effective popularization, at an extremely low price, of the techniques and state of the Soviet digital (and to a lesser extent, analog) computer art. Of particular interest is the inclusion of the use of such machines under the heading of "cybernetics", which scientists in the USSR have taken from Norbert Wiener's book of the same name.

After an introduction and definition of cybernetics, the book discusses information theory and its applications, the transformation of information, and self-regulating systems. The construction and design of computers is represented by the MN-2 Analog Computer and the Strela, BESM, and URAL digital computers. An introduction to the basis of digital computer programming makes use of the Strela instruction code, and relative address coding.

In the discussion of digital computers there are brief accounts of Soviet work on automatic programming, machine translation, and game-playing programs. The book predicts a wide extension of the use of such computers as aids to scientists' and engineers' difficult mental tasks, as well as the advent of the automatic factory. It is unfortunate that the authors' choice of examples of U.S. work in such areas were of the more flamboyant type, but this can probably be blamed on lack of communication.

At present there appears to be no comparable popular book on computers of this recent vintage in Western Europe or the U.S.A., except for F.-H. Raymond's similar book [*L'automatique des informations*, Masson, Paris, 1957; MR **20** #4357].

J. W. Carr, III (Chapel Hill, N.C.)

3946:

Gelernter, H. A note on syntactic symmetry and the manipulation of formal systems by machine. Information and Control **2** (1959), 80–89.

In order to reduce the amount of exploration in a computer program for proving theorems in Euclidean geometry, the author uses a concept of syntactic symmetry of a figure with respect to the hypotheses of a theorem, and also introduces the related notion of syntactic conjugate of a formula. The program may assume any syntactic conjugate of any formula it has proved and should not set up as a sub-goal a syntactic conjugate of any sub-goal it already has since the latter will be equally difficult to prove. The author describes a procedure for recognizing symmetries.

J. McCarthy (Cambridge, Mass.)

3947:

Èl'yasberg, V. M. Study of algebraic equations on analog computers. Avtomat. i Telemeh. **20** (1959), 756–761. (Russian. English summary)

A simple method of solution of algebraic equations on analog computers is described. The method is based on plotting certain functions as a result of solving determinative differential equation and allows one to find real and imaginary roots of n-order polynomial. There are several examples for $n\leq 6$. *Author's summary*

3948:

Kryže, Irži. Device for solving high-order algebraic equations. Avtomat. i Telemeh. **20** (1959), 762–772. (Russian. English summary)

A device for reproducing algebraic polynomials and for solving high-order equations is described. The construction is based upon a new principle that allows its full automatization with minimum number of electronic tubes. The part of the computer where polynomials are generated has no electronic tubes at all.

The repetition rate of the solutions is equal to 10–50 times per second; therefore it is possible to follow, equation roots change permanently when equation coefficients change continuously. The probable sources of the computer errors are considered. *Author's summary*

MECHANICS OF PARTICLES AND SYSTEMS

See also 3924, 3938, 4037, 4038, 4039.

3949:

Onicescu, Octav. La mécanique de certaines particules stables. Rend. Sem. Mat. Univ. Padova **28** (1958), 322–330.

The author studies the motion of a small rigid body in an inertial system and in a field by means of a set of postulates for the motion of a particle set forth in an earlier paper [J. Math. Mech. **7** (1958), 723–740; MR **21** #451]. *H. Tornehave* (Copenhagen)

3950:

Lipka, István. Beitrag zur Arbeit "Die mathematische Lösung eines Werkzeuggeometrischen Problems" von Jr. I. Drahos, L. Hornyik und M. Hosszú. Magyar Tud. Akad. Mat. Kutató Int. Közl. **3** (1958), 219–236. (Hungarian. Russian and German summaries)

The paper referred to is Magyar Ind. Akad. Mat. Kutató Int. Közl. **3** (1958), 83–96 [MR **21** #988].

3951:

Agostinelli, Catoldo. On the rolling of a non-homogeneous disc. Ci. y Téc. Buenos Aires. **127** (1959), 164–169. (Spanish. Italian and English summaries)

This article raises the issue of the equations of the rolling movement of a heavy non-homogeneous disc on an horizontal plane supposing that the lack of homogeneousness is generated by a small mass situated in the plane of the disc. *Author's summary*

3952:

Marsicano, Fénix Roberto. On the motion of a point of variable mass attracted to a fixed center by newtonian force. Ci. y Téc. Buenos Aires. **127** (1959), 170–182. (Spanish. English summary)

The paper deals with a planetary movement of variable mass. Perturbation theory is applied. A time linear function for mass variation is assumed in integrating equations. *Author's summary*

3953:

Tatarkiewicz, Krzysztof. Un exemple simple de mouvement non holonome. Ann. Univ. Mariae Curie-Skłodowska. Sect. A **11** (1957), 5–16 (1959). (Polish and Russian summaries)

Un point matériel des coordonnées (x_1, x_2, x_3) est assujetti à la liaison

$$\sum a_{\nu\mu}\dot{x}_\nu\dot{x}_\mu = 0,\ \nu, \mu = 0, 1, 2, 3,\ \dot{x}_0 = 1,\ a_{\nu\mu}(x_1, x_2, x_3, t),$$

qui est une liaison non holonome si $\sum a_{\nu\mu}y_\nu y_\mu = 0$ représente une surface du second degré qui n'est pas dégénérée. L'auteur considère le cas particulier $\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2=v^2$ où v est une constante et analyse les réactions nécessaires. Un exemple est la descente d'un alpiniste sur un plan incliné sous l'influence de la pesanteur et du frottement. Le système des équations de mouvement se résolut par des fonctions élémentaires, le trajectoire étant une courbe transcendante avec deux asymptotes verticales. *O. Bottema* (Delft)

3954:

Minorsky, Nicolas. Nouvelles méthodes de la théorie des oscillations. Confer. Sem. Mat. Univ. Bari **25-26-27-28**, 64 pp. (1957).

This monograph by the well-known author of *Introduction to nonlinear mechanics* [Edwards, Ann Arbor, Mich., 1947; MR 8, 583] consists of four lectures given at the University of Bari. In the main it is more directed to physicists and engineers than to mathematicians. The mathematically inclined will however peruse with profit these lectures to see how mathematics is, and can be used in the "outside". The various topics are as follows.

First lecture: Topological methods (essentially Poincaré's theory of limit-cycles).

Second lecture: Analytical methods: Poincaré's expansion theorem, approximation methods of van der Pol and Krylov-Bogoliubov.

Third lecture: The author's stroboscopic method: study of the neighborhood of a periodic solution by means of the points where a near-by solution pierces a fixed section.

Fourth lecture: Relaxation oscillations. *S. Lefschetz* (Princeton, N.J.)

3955:

Gold, Louis. A reduced solution for the simple pendulum: New approach to elliptic functions. J. Franklin Inst. **267** (1959), 503–509.

The equation $d^2\varphi/dt^2 = -\sin\varphi$ is "solved" by expanding t in powers of φ. The general term in this expansion is not given. *P. Henrici* (Los Angeles, Calif.)

STATISTICAL THERMODYNAMICS AND MECHANICS

See also 4020, 4030.

3956:

Reiss, H.; Frisch, H. L.; and Lebowitz, J. L. Statistical mechanics of rigid spheres. J. Chem. Phys. **31** (1959), 369–380.

The statistical thermodynamics of a fluid of rigid spheres is investigated by the method which introduces an additional sphere of arbitrary radius into a system of otherwise identical rigid spheres. By equating two different expressions for the pressure, the authors obtain an integral condition which is satisfied by the pair distribution function. This is supplemented by certain other conditions and used to determine the pair distribution function in an assumed form with four arbitrary parameters; the results are in satisfactory agreement with those obtained previously by other methods. The first few virial coefficients are computed. The values obtained are then compared with those obtained by a different, though somewhat similar method. The surface tension is computed, and is negative as would be expected in the absence of attractive forces. *H. S. Green* (China Lake, Calif.)

3957:

Maslennikov, M. V. On Wick's problem. Soviet Physics. Dokl. **120** (3) (1958), 521–525 (59–62 Dokl. Akad. Nauk SSSR).

The problem in the title concerns itself with the slowing down and diffusion of neutrons by a plane source in an infinite homogeneous isotropic medium. While this has been studied by several authors [see for example G. C. Wick, Phys. Rev. (2) **75** (1949), 738–756] who developed asymptotic formulae for the spatial and energy distribution of the density of neutron impacts, the author of this paper sketches a rigorous mathematical solution of this problem. His starting point, as with other authors, is in using the Fourier-Laplace transformation of the collision density. While avoiding continued fractions, the author uses powerful tools, such as the Krein-Rutman theory of invariant cones in a Banach space and the method of steepest descent in evaluating certain integrals, to obtain the desired asymptotic formula. *R. S. Varga* (Pittsburgh, Pa.)

3958:

Bellman, Richard; Kalaba, Robert; and Wing, G. Milton. Invariant imbedding and neutron transport theory. III. Neutron-neutron collision processes. J. Math. Mech. 8 (1959), 249–262.

[For parts I and II, see same J. 7 (1958), 149–162, 741–756; MR 20 #2046, #6938.] The authors develop further applications of their method of invariant embedding (see part I). Idealized one-dimensional ("slab geometry")

problems are treated, in which "neutrons" can be scattered, absorbed, and allowed to produce new neutrons. Such problems are reduced to initial value problems for ordinary differential equations, in the case of one type of particle. If two types ("fast" and "slow" neutrons) are considered, partial differential equations ensue. Some numerical results are given for the case of one velocity-group. *G. Birkhoff* (Cambridge, Mass.)

3959:

Bellman, Richard; Kalaba, Robert; and Wing, G. Milton. Invariant imbedding and neutron transport theory. IV. Generalized transport theory. J. Math. Mech. **8** (1959), 575–584.

The idea of invariant embedding (see the preceding review) is applied to one-parameter families of surfaces $S(\eta)$ in space, depending on a parameter η. The probability of being reflected from $S(\eta)$ is shown to satisfy an integro-differential equation and a stochastic functional equation. Previous treatments by the authors (see part II) of idealized neutron problems in cylinders and spheres, are interpreted as applications of this new formalism.

G. Birkhoff (Cambridge, Mass.)

3960:

Zubarev, D. N.; and Tserkovnikov, Iu. A. On the theory of the phase transition in a nonideal Bose gas. Soviet Physics. Dokl. **120 (3)** (1958), 603–607 (991–994 Dokl. Akad. Nauk SSSR).

The authors adopt for the imperfect Bose gas a truncated hamiltonian containing only two types of collisions: exchange collisions where the incoming particles simply take over each other's states, and collisions of pairs of particles with opposite momenta. Treating further the occupation number of the zero momentum state as macroscopic and unquantized, one can calculate the grand partition function exactly in the limit of a large system. To lowest order in the interaction (the only order in which the truncated hamiltonian can be expected to give correct results) one finds simple results both below and above the transition temperature.

L. Van Hove (Utrecht)

ELASTICITY, PLASTICITY

See also 3643, 3679.

3961:

★Müller, Wilhelm. Theorie der elastischen Verformung. Mathematik und ihre Anwendungen in Physik und Technik. Reihe A, Bd. 27. Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1959. xi+327 pp. DM 31.50.

An introductory text on elasticity theory including some topics in the theory of plates and shells, in instability theory, and in vibration theory.

E. Sternberg (Providence, R.I.)

3962:

Vlasov, V. V. Application of the method of initial functions to a plane problem of the theory of elasticity for a rectangular region. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. **1959**, no. 3, 114–125. (Russian)

The equilibrium of a rectangular strip ($0 \leq x \leq l$, $0 \leq z \leq h$) under the conditions of plane stress is treated by the method of "initial functions" [V. Z. Vlasov, same Izv. **1955**, no. 7, 49–69; MR **17**, 557]. According to this procedure the components of the stress and displacement at any point in the strip are given as the result of the operation of certain transcendental differential operators upon the components of stress and displacement along an "initial" boundary. In this paper the initial boundary comprises the two edges $z=0$ and $z=h$, and the semi-inverse method is used to solve the problem formally when on these edges (i) $u=Z_z=0$, (ii) $X_z=Z_z=0$, (iii) $u=w=0$ are assumed in turn (standard notation). It turns out that these assumptions in each case permit arbitrary (i.e., elastically admissible) boundary conditions to be imposed on $x=0$ and $x=l$. *R. N. Goss* (San Diego, Calif.)

3963:

Nikitin, Yu. P. Method of nomographic determination of curved axes of flexible bars and shells. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 718–723. (Ukrainian. Russian and English summaries)

A nomographic method is considered for solving problems on the determination of large elastic displacements of flexible bars, under the effect of concentrated forces and moments, and problems on the choice of rational axes of structures under the effect of hydrostatic loads. The solution is found with the aid of two lattice nomograms employed independently of each other. The nomograms were constructed on the basis of Euler's integral curves of the exact equation of bending. *Author's summary*

3964:

Burmistrov, E. F. Some problems in the theory of bending of a thin isotropic slab with a hole of arbitrary shape. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 9, 143–147. (Russian)

In this paper the problem of concentration of strain in a flexible thin isotropic slab with a hole of arbitrary shape is investigated.

For the solution of the first and the second fundamental problems of the theory of elasticity the author uses the Mushelišvili method [*Nekotorye osnovnye zadači matematičeskoĭ teorii uprugosti*, 4th ed., Izdat. Akad. Nauk SSSR, Moscow, 1954; MR **16**, 1067] as well as some results obtained previously by S. G. Lechnizky [Akad. Nauk SSSR Prikl. Mat. Meh. **2** (1938), 181–210].

The author states that from his general solution it is possible to derive all results on this question obtained till now [M. M. Friedman, ibid. **9** (1945), 334–338; MR **8**, 116; G. N. Savin, *Koncentraciya napryaženiĭ okolo otverstiĭ*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951; MR **15**, 370]. Beyond that, it is demonstrated that some results obtained by G. N. Savin in the mentioned monograph were false.

Four particular cases of load are considered at the end of the paper. *T. P. Anđelić* (Belgrade)

3965:

Makai, E. Bounds for the principal frequency of a membrane and the torsional rigidity of a beam. Acta Sci. Math. Szeged **20** (1959), 33–35.

Let Λ be the principal frequency and P the torsional rigidity of a plane domain D with area A and perimeter L. The author proves the inequalities $\Lambda \leqq 3^{1/2}LA^{-1}$ and $P \geqq L^{-2}A^3$. The proof is carried through for polygons and then extended to general domains by continuity.

While the paper was in print, G. Pólya, in a paper to be published in the J. Indian Math. Soc., improved the above inequalities to $\Lambda \leqq \frac{1}{2}\pi LA^{-1}$ and $P \geqq \frac{1}{3}L^{-2}A^3$, which approach equality as D approaches an infinite strip.

H. F. Weinberger (College Park, Md.)

3966:

Reissner, Eric. On the determination of stresses and displacements for unsymmetrical deformations of shallow spherical shells. J. Math. Phys. **38** (1959/60), 16–35.

This paper is concerned with problems of bending of thin, elastic, shallow spherical shells of uniform thickness. It supplements earlier work of the author.

Explicit expressions are derived, in terms of stress functions and axial displacement components, for radial and circumferential displacement components. It is shown that certain one-valued portions of the solution for stress functions and axial displacement components give rise to multi-valued expressions for radial and circumferential displacement components.

A new type of solution of the differential equations is derived which is many-valued insofar as the stress function is concerned but gives one-valued expressions for all quantities which should be one-valued. It is shown that this type of solution is needed for the analysis of problems for which a resultant side force is acting along the edge of a symmetrical circular boundary of the shell. The nature and form of this side force solution differs in important respects from the corresponding known solution for the case of a flat plate.

As a specific application, a shell is considered which is supported at its outer edge and which has a small symmetrical rigid insert. The rigid insert is acted upon by a tilting moment and by a side force. Determined in particular are the angular displacement and the lateral displacement of the insert in terms of the moment side force.

A further general result consists of a series solution for the effect of distributed axial surface loads. The first term of this series corresponds to the result of membrane theory for the distributed surface load problem.

As an application of the surface load solution, the membrane stresses and deformations caused by a lateral hydrostatic loading are determined.

Consideration is given to the problem of interior versus boundary layer stress states in a shell segment with prescribed edge displacement. Conditions are established for the allowable magnitude of normal edge displacement and slope, in terms of meridional and circumferential edge displacements, such that the interior state depends solely on meridional and circumferential edge displacements. It is further established under what conditions the interior state is a membrane state, state of inextensional bending, or a mixture of the two. *W. Zerna* (Hanover)

3967:

Kupradze, V. D. On boundary problems in the theory of elasticity for piece-wise inhomogeneous bodies. Soobšč. Akad. Nauk Gruzin. SSR **22** (1959), 129–136. (Russian)

An elastic medium B_a characterized by Lamé constants λ_a and μ_a, occupying an infinite space and possessing an arbitrary elastic inclusion B_i with the boundary S characterized by λ_i and μ_i is considered. The medium is subjected to an action of source P_0 of periodic vibrations of constant strength and frequency ω.

It is shown that if the problem possesses a solution in the class of vectors admitting an application of Betti theorem, the solution satisfies the following equations:

$$(*)\quad \binom{\mu_i}{\mu_a}u_k(P) = \frac{1}{4\pi}\omega^2\int_{(B_i)} u(Q)\Big[(\mu_a-\mu_i)\Gamma^k(P,Q) - \mu_a\binom{\tau}{\tau_1}\Gamma^{k(n)}(P,Q)\Big]d\tau_Q \mp \frac{1}{4\pi}\mu_a\binom{\tau}{\tau_1}\int_{(B_i)} u(Q)\,\mathrm{grad}\,\frac{\partial}{\partial\xi_k}\frac{1}{\tau}\,d\tau_Q + \frac{\mu_a}{4\pi}\int_{(S)} u_i(Q)T^*\binom{\Gamma_*{}^k(P,Q)}{\Gamma^k(P,Q)}ds_Q + \binom{\mu_a F_k(P,P_0)}{\mu_a E_k(P,P_0)},$$

the upper and lower rows corresponding to $P\in B_i$ and $P\in B_a$, respectively. Notation: u = displacement vector ($k=1, 2, 3$); Γ^k and $\Gamma^{k(n)}$ = solutions of $\mu_a\Delta u+(\lambda_a+\mu_a)$ grad div $u+\omega^2u=0$ and $=$ grad $(\partial/\partial x_k)(1/r)$, respectively; $\Gamma_*{}^k$ a linear combination of Γ^k and $\Gamma^{k(n)}$; $\tau=\alpha/(\lambda_i+2\mu_i)$, $\tau_1=\alpha/(\lambda_a+2\mu_a)$ with $\alpha=(\lambda_i\mu_a-\lambda_a\mu_i)/\mu_a$; $T^*=T^a-T^i$ where T^i and T^a = limit values on S of stress vectors in B_i and B_a, respectively; F_k = a given function and E_k = the displacement in B_a in the absence of the inclusion B_i. {Many misprints make the reading of the paper difficult and cast a shadow on the correctness of eqs. (*).}

J. Nowinski (Madison, Wis.)

3968:

Stavsky, Yehuda. On a cross-elasticity phenomenon in symmetrically nonhomogeneous plates. J. Aero/Space Sci. **26** (1959), 607.

3969:

Keller, Joseph B. Large amplitude motion of a string. Amer. J. Phys. **27** (1959), 584–586.

The motion of a string is investigated without assuming that the amplitude of the motion is small or that the motion is purely transverse. By assuming a special stress-strain law for the string material, we find that the three components of motion are independent and that each component satisfies the usual linear wave equation. The derivation of these exact results seems to be simpler than the usual approximate derivations. It is also shown that the special stress-strain law employed is the only one for which purely transverse motion is possible.

Author's summary

3970:

Federhofer, Karl. Nicht-lineare Biegungsschwingungen des Kreisringes. Ing.-Arch. **28** (1959), 53–58.

3971:

Mathews, P. M. Vibrations of a beam on elastic foundation. II. Z. Angew. Math. Mech. **39** (1959), 13–19. (German, French and Russian summaries)

The author extends his earlier work [same Z. **38** (1958), 105–115; MR **20**, 512] to include the effects of damping on the vibrations of a beam on an elastic foundation subject to a load $F_0 \cos \omega t$ moving at a constant velocity v along the beam. A Fourier transformation of the differential equation leads to a fourth degree polynomial equation in the transform variable whose coefficients involve v, ω and the damping coefficient ν. The author notes the difficulty of solving this equation in general, and, while pointing out the possibility of a solution for any given values of v, ω, and ν, limits his explicit solutions to the following cases: (i) alternating load at fixed point ($v=0$; ω, $\nu\neq 0$); (ii) moving load of constant magnitude ($\omega=0$; v, $\nu\neq 0$). In these cases the solution is presented in graphical and tabular form; the author further notes that his solution in case (ii) agrees with that of Kenney [J. Appl. Mech. **21** (1954), 359–364].

W. E. Boyce (Troy, N.Y.)

3972:

Kisilivs'ka, L. M. Free oscillations of a shell reinforced with stiffening ribs. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 730–735. (Ukrainian. Russian and English summaries)

A general method is described for determining the frequencies of free oscillation of a shell with an arbitrary inner surface, reinforced by stiffening ribs placed along the lines of curvature of the inner surface. This method is developed in detail for a sloping circular cylindrical shell, reinforced by longitudinal equidistant stiffening ribs.

From the author's summary

3973:

Cox, Hugh L.; and Benfield, Wilcomb, A. Vibration of uniform square plates bounded by flexible beams. J. Acoust. Soc. Amer. **31** (1959), 963–966.

The fundamental frequencies of flexural vibration are determined for uniform isotropic square plates that have pinpoint supports at the four corners and flexible beams along the edges. The flexible beams are pinned to the plate so that the beams resist only deflection of the plate boundaries. Finite difference solutions, which simplify the treatment of the boundary conditions for definite values of Poisson's ratio, are used to obtain the approximate solutions.

Author's summary

3974:

Fischer, H. C. On longitudinal impact. II. Elastic impact of bars with cylindrical sections of different diameters and of bars with rounded ends. Appl. Sci. Res. A **8** (1959), 278–308.

The author applies the graphical method he has developed in a previous paper [Appl. Sci. Res. A **8** (1959), 105–139; MR **20** #7436] to treat the elastic impact of a cylindrical hammer on the end of a bar, where the bar and hammer are of different diameters. The method is also applied to the impact of bars with rounded ends and it is shown that the graphical treatment is equivalent to the iteration method used by Sears in his classical work on this problem. Comparison is made between the theoretical predictions and experimental results in a number of cases, and fairly good agreement is obtained.

H. Kolsky (Fort Halstead)

3975:

Savin, G. M.; Goroško, O. O.; and Bezsonov, V. G. Determination of stresses in a reeling elastic rope. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 712–717. (Ukrainian. Russian and English summaries)

The authors investigate the stress distribution in the reeling parts of ropes. The equilibrium conditions for the thread on the felloe are determined and it is shown that, at winding-up speeds of v_0 = constant, the dynamic stresses in the reeling part are almost completely damped by friction forces.

From the authors' summary

3976:

Sveklo, V. A.; and Siukiiainen, V. A. Diffraction of a plane elastic wave at the vertex of a sector. Soviet Physics. Dokl. **119 (3)** (1958), 451–452 (1122–1123 Dokl. Akad. Nauk SSSR).

3977:

Rivlin, R. S. The constitutive equations for certain classes of deformations. Arch. Rational Mech. Anal. **3**, 304–311 (1959).

The author presents arguments indicating that time dependent stress-strain laws of the type used in nonlinear elasticity theory are adequate to describe a special class of deformations in more complex solids. For these, the displacement is of the form $u_i = f(t)v_i$, where $f(t)$ is a specified function of time and v_i is any function of position. For viscoelastic materials, the form of the stress-strain relations will be different for different choices of $f(t)$. An analysis of combined extension and torsion of a right circular cylinder is given. He points out some special stress-strain laws for which the equations of equilibrium with zero body force are satisfied at all times if they are satisfied at one.

J. L. Ericksen (Baltimore, Md.)

3978:

Hill, R. Some basic principles in the mechanics of solids without a natural time. J. Mech. Phys. Solids **7** (1959), 209–225.

The paper is concerned with the isothermal deformation of solids in which a typical component of the rate of stress depends on the components of stress, variables representing the strain history, and the components of the rate of deformation in a manner that is homogeneous of degree one though not necessarily linear in the latter. These solids are classified according to their behavior in infinitesimal cycles of deformation.

A typical boundary value problem is formulated and a sufficient condition for the uniqueness of its solution is established. Stability under dead loads is investigated. For solids in which the components of the rate of stress are the partial derivatives of a homogeneous function of degree two in the velocity gradients, stronger results concerning uniqueness and stability are obtained.

Some definitions of stress-rate are discussed, in particular the convected derivative of the Kirchhoff stress, which is used to construct constitutive equations for elastic, plastic solids.

W. Prager (Providence, R.I.)

3979:

Marin, Joseph. Creep stresses and strains in an axially loaded plate with a hole. J. Franklin Inst. **268** (1959), 53–60.

3980:

Danilovskaya, V. I. Temperature field and temperature stresses arising in an elastic half-space as a result of a flow of radiative energy falling on the boundary of the half-space. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. **1959**, no. 3, 129–132. (Russian)

A flow of radiative energy of intensity $I(t)=I_0$ for $t\leqq t_0$ and $I(t)=0$ for $t>0$ falling on the boundary of the half-space D is considered. It is assumed that: (1) the capability of absorption of the energy in the half-space decreases exponentially with depth x, (2) the initial temperature throughout D is zero, (3) there is an exchange of heat between D and the outer medium of zero temperature, (4) the lateral displacements v and w vanish at infinity, (5) the boundary of D is free from external tractions.

The temperature and stress fields are found using the operational calculus. *J. Nowinski* (Madison, Wis.)

3981:

Mossakowski, Jerzy. Thermal stresses in an elastic space with discontinuous physical properties. Arch. Mech. Stos. **11** (1959), 243–258. (Polish and Russian summaries)

Consider two semi-infinite elastic bodies, each bounded by a plane, perfectly bonded along their common boundary. Let the two bodies have distinct thermal and elastic properties. The author determines the steady-state temperature and stress distribution induced in both media by a point source of heat located at an interior point of one of them. Numerical examples are included.

E. Sternberg (Providence, R.I.)

3982:

Nowacki, Witold. Some dynamic problems of thermoelasticity. Arch. Mech. Stos. **11** (1959), 259–283. (Polish and Russian summaries)

Formal solutions of the coupled thermo-elastic field equations are obtained corresponding to various harmonically oscillating heat and temperature sources in a medium occupying the entire space or a half-space. Point-sources, line-sources, and plane surface-sources are considered. The coupled problem associated with an oscillatory concentrated load at a point of a body of unlimited extent is also discussed. *E. Sternberg* (Providence, R.I.)

3983:

Derski, Wlodzimierz. Non-steady-state of thermal stresses in a layered elastic space with a spherical cavity. Arch. Mech. Stos. **11** (1959), 303–316. (Polish and Russian summaries)

Let r be the distance from the origin and let each of the regions $a\leqq r\leqq b$ ($a>0$, $b>a$) and $b\leqq r<\infty$ be occupied by a homogeneous and isotropic elastic solid. Suppose the two materials to have distinct thermal and elastic properties and to be perfectly bonded at the interface $r=b$. The author obtains a solution in integral form for the thermal stresses induced in the composite body by a uniform, time-dependent distribution of the surface temperature applied to the internal boundary $r=a$. Inertia effects and thermoelastic coupling are disregarded.

E. Sternberg (Providence, R.I.)

3984:

Nowacki, Witold. Two one-dimensional problems of thermoelasticity. Arch. Mech. Stos. **11** (1959), 333–346. (Polish and Russian summaries)

This paper deals with the determination of the thermal stresses in a half-space $x\geqq 0$, arising from heat sources which are uniformly distributed over the plane $x=\xi>0$ and are a step function of time; the boundary $x=0$ is insulated. The foregoing problem is treated by means of the Laplace transform. Both a homogeneous, isotropic, linear elastic and a visco-elastic medium are considered. Inertia effects are taken into account, thermo-mechanical coupling is disregarded, and all material properties are assumed to be independent of the temperature.

E. Sternberg (Providence, R.I.)

STRUCTURE OF MATTER

3985:

Merrifield, R. E. Interaction of excitation waves in a one-dimensional crystal. J. Chem. Phys. **31** (1959), 522–525.

The problem of exciton-exciton interaction in molecular crystals is discussed in terms of the stationary two-quantum excited states of an idealized one-dimensional crystal. If only a single molecular excited state is considered, the wave functions and energy levels of the two-quantum states may be obtained directly from those of the corresponding one-quantum problem with the proviso that the two excitons have different wave numbers. If in addition, a second molecular excited state is included, there can exist states in which two excitons remain bound to each other as they propagate through the crystal.

Author's summary

FLUID MECHANICS, ACOUSTICS

See also 3758, 4047, 4048.

3986:

Aymerich, Giuseppe. Teoria linearizzata di una classe di moti piani non stazionari di un gas perfetto. Rend. Sem. Fac. Sci. Univ. Cagliari **28** (1958), 33–39.

The author continues his analysis of non-stationary plane motions of a perfect gas characterized by the invariance of the lines of flow with respect to the time [same Rend. **27** (1957), 27–34; MR **20** #2138]. A perturbation calculation leads to an equation of Darboux. The application of the linearized theory to the problem of a thin profile connects the analysis with Abel's integral equation.

C. D. Calsoyas (Livermore, Calif.)

3987:

Melis, Antonio. Su una classe di moti piani non stazionari di fluidi comprimibili con linee di flusso permanenti. Rend. Sem. Fac. Sci. Univ. Cagliari **28** (1958), 40–47.

The author applies a perturbation method to Aymerich's analysis [same Rend. **27** (1957), 27–34; MR **20** #2138] of

non-stationary plane motions of a perfect gas characterized by the invariance of the lines of flow with respect to time. Particular cases are treated.

C. D. Calsoyas (Livermore, Calif.)

3988:

Rosenblat, S. The aerodynamic forces on an aerofoil in unsteady motion between porous walls. Quart. J. Mech. Appl. Math. **12** (1959), 151–174.

A solution is obtained to the boundary-value problem arising in the unsteady motion of a thin aerofoil in a stream bounded by porous walls. The flow is two-dimensional, inviscid, and incompressible. The condition holding at the porous boundaries is assumed to be a proportionality between the normal velocity component at the wall and the pressure difference across the wall. The solution relies on a form of this boundary condition which is valid for small values of the frequency of the oscillations, although conditions at the aerofoil surface are independent of frequency. The usual assumption of small amplitude of the disturbances is made. A formula is derived for the pressure on the aerofoil surface in terms of the flow direction at the surface. The relations for the lift and moment about the mid-chord point are obtained for a harmonic upwash distribution. For the case of a rigid aerofoil, particular types of harmonic oscillations are investigated, and the lift and moment are given as dimensionless 'air-load coefficients'. The results are expressed as the first-order terms of expansions in powers of a parameter which depends on the ratio of aerofoil chord to tunnel width, and are valid for small values of this ratio.

G. N. Lance (Southampton)

3989:

Chandrasekhar, S. The oscillations of a viscous liquid globe. Proc. London Math. Soc. (3) **9** (1959), 141–149.

The normal modes of oscillations of an inviscid liquid of spherical form can be characterized by the order l of the spherical harmonic which describes the deformed surface. This paper is concerned with how these modes are damped by viscosity. It is assumed that the deformed surface is of the form $r=R+\varepsilon_0 P_l^m(\theta)\exp(-\sigma t\pm im\varphi)$, where R is the radius of the unperturbed sphere and σ is to be determined. Solving the disturbance equations and applying the conditions that the tangential stresses vanish at the undeformed surface and the radial stress vanish at the deformed surface gives a complicated transcendental equation for σ. For $\alpha^2=\sigma_{l;0}R^2/\nu$ ($\sigma_{l;0}$ is the circular frequency of the lth inviscid mode; ν is the kinematic viscosity) less than a determined $\alpha^2_{\max}$ the decay is aperiodic. In this case there are two principal modes of decay which are computed for varying α^2 for $l=2$, 3 and 4. For $\alpha^2\geqq\alpha^2_{\max}$ the lowest modes of decay are of an oscillatory nature.

R. C. DiPrima (Troy, N.Y.)

3990:

Squire, William. Application of generalized Gauss-Laguerre quadrature to boundary-layer problems. J. Aero/Space Sci. **26** (1959), 540–541.

3991:

Gupta, A. S. Effect of buoyancy forces on certain viscous flows with suction. Appl. Sci. Res. A **8** (1959), 309–320.

Some particular cases of the two-dimensional flow of a viscous liquid past an inclined flat plate at a maintained temperature are considered; in the first problem, for instance, there is a constant wall suction and the velocity of the plate is an exponential function of the time. Boundary layer equations are used, and the velocity and temperature distributions are given by simple functions.

W. R. Dean (London)

3992:

Roy, Ajit Kumar. Estimation of the critical viscous sub-layer in shock wave boundary layer interaction. Z. Angew. Math. Phys. **10** (1959), 82–89. (German summary)

The reviewer's theory [Proc. Roy. Soc. London Ser. A **217** (1953), 478–507], of shock-wave-boundary-layer interactions without separation made use of the fact that the viscous terms in the equation for the disturbance to the boundary layer are significant only in a thin inner sub-layer, where speeds are so low that compressibility can be neglected. In this paper the thickness of the inner sub-layer is explicitly calculated for a number of cases of both laminar and turbulent flow, and found to be between 1/10 and 1/100 of the boundary layer thickness, in agreement with the reviewer's theory.

M. J. Lighthill (Manchester)

3993:

Serrin, James. On the stability of viscous fluid motions. Arch. Rational Mech. Anal. **3** (1959), 1–13.

The energy method of Reynolds and Orr is used to derive universal criteria for the stability of arbitrary fluid motions. The principal result obtained is that, for an arbitrary flow in a bounded region, a sufficient condition for stability is that the Reynolds number (based on the maximum velocity of the basic flow and the diameter of the region) be less than 5.71. The method is also applied to Couette flow, for which a somewhat sharper estimate can be obtained. In this case the flow is stable with respect to arbitrary disturbances whenever

$$\left|\frac{\Omega_2-\Omega_1}{\nu}\right| < (R_2^2-R_1^2)\left\{\frac{\pi}{R_1R_2\log(R_2/R_1)}\right\}^2.$$

W. H. Reid (Providence, R.I.)

3994:

Serrin, James. A note on the existence of periodic solutions of the Navier-Stokes equations. Arch. Rational Mech. Anal. **3** (1959), 120–122.

The existence of stable periodic solutions of the Navier-Stokes equation is established for a bounded region of space V (which may depend on the time periodically) on the boundary of which a period velocity is prescribed, provided two conditions are satisfied: (i) for prescribed initial velocities over V there corresponds a solution of the Navier-Stokes equations satisfying the prescribed boundary conditions and (ii) there is one solution whose Reynolds number is less than 5.71 [see the previous review].

W. H. Reid (Providence, R.I.)

3995:

Hassan, H. A. Some aspects of three-dimensional diabatic flows. Proc. Iraqi Sci. Soc. **2** (1958), 11–14. (Arabic summary)

The author considers a diabatic flow, i.e., a flow of an

inviscid, non-heat conducting fluid with heat addition by means of sources. In particular, the author deals with the variation of pressure, density, temperature and vorticity along streamlines, their principal normals and binormals. The fundamental equations are those of the classical hydrodynamics plus the first law of thermodynamics. The geometrical relations between the unit vectors along the tangent, the principal normal and the binormal are derived next. The operations which follow are: derivation of the generalized Bernoulli equation and elimination of the density from the momentum equation. The following variations are considered: of pressure, density, temperature and vorticity. The pressure remains constant along the binormals to the streamlines. This is also true in the absence of heat addition. The density and temperature variations along binormals are due to heat addition only. The variation of vorticity does not contribute anything new. The present review would be incomplete without mentioning a slight deficiency in the paper. The idea of using the total amount of heat added to a particle between some initial point and the point in question along a streamline was communicated to the author by the reviewer. But this is closely related to the fundamental problem of interpreting the amount of heat in the first law of thermodynamics as the total differential. This was a very long controversy in physics solved by K. Menger [Amer. J. Phys. **18** (1950), 89–103] and the reviewer [Acta Phys. Austriaca **12** (1958), 60–69; MR **19**, 1229]. Without these explanations the paper is incomplete. The conclusion (referred to author's thesis in 1955) that in an irrotational diabatic flow the Bernoulli constant must be a universal constant is a general law of the irrotational flow and as such is very old and was merely rediscovered by the author in 1955.

M. Z. v. Krzywoblocki (Urbana, Ill.)

3996:

Sakurai, Takeo. High subsonic flow with normal shock wave at nearly critical Mach number. J. Phys. Soc. Japan **14** (1959), 658–663.

A simple approximate solution is presented for the supercritical transonic flow past an aerofoil. The flow equations are approximated by the subsonic linearized equation in the subsonic portion of the flow while in the imbedded supersonic region a linear equation in which both families of Mach lines are normal to the main stream is used. The boundary between these two regions is determined by the condition that the solutions in the separate regions should match on the boundary. The calculations appear easy and to agree with more exact theories.

J. J. Mahony (Sydney)

3997:

Aymerich, Giuseppe. Su le onde di rarefazione nei canali e nei tubi di sezione poco variabile. Boll. Un. Mat. Ital. (3) **13** (1958), 543–550.

The author considers small linearized unsteady perturbations of steady uniform flow. For the plane case let x, y be rectangular coordinates relative to axes uniformly translated parallel to the x-axis. Let θ be a linear function of x/t with coefficients that are known functions of γ and the undisturbed speed of sound. Let $\xi=\theta^{-2/\alpha}$, $\eta=x^2\theta^{2/\alpha}$, and $f(\xi,\eta,y)=\xi^{-0.5}\phi(x,\theta,y)$, where $\alpha=2(\gamma-1)/(3-\gamma)$ and ϕ is the perturbation velocity potential. Then f satisfies $4\partial^2 f/\partial\xi\partial\eta-\partial^2 f/\partial y^2=0$ with $f\hat{y}(\xi,\eta,0)=0$ and $f\hat{y}(\xi,\eta,b)$ known at the wall $y=b$. In the Fourier series expansion $f=\sum p\hat{n}(\xi,\eta)\cos n\pi y/b$, the coefficients satisfy

$$(*)\qquad \partial^2 p\hat{n}/\partial\xi\partial\eta+(n\pi/2b)^2 p\hat{n} = 0.25(-)^n f\hat{y}(\xi,\eta,b)$$

and can be found by quadratures with the aid of the known Riemann function of (*). The determination of coefficients in the Fourier-Bessel expansion of the corresponding axisymmetric perturbation potential follows the identical pattern. For a step discontinuity at the wall the functions $p_0(\xi,\eta)$ correspond to results previously obtained for one-dimensional unsteady flow by W. Chester [Quart. J. Mech. Appl. Math. **7** (1954) 57–82; MR **16**, 185].

J. H. Giese (Aberdeen, Md.)

3998:

Sakurai, Takeo. On the normal shock wave attached to the curved surface. J. Aero/Space Sci. **26** (1959), 460–461.

The author points out a discrepancy between two published results on the existence of a normal shock wave attached to a convex surface in two-dimensional compressible flow. He shows that the difference is due to an incorrectly stated boundary condition in one paper and that when this is corrected both theories predict that no such shock wave can exist in the Mach number range 1 to 1.662.

J. J. Mahony (Sydney)

3999:

Charrueau, André. Sur les ondes de choc. Ann. Ponts Chaussées **129** (1959), 497–549. (English summary)

This is a detailed study of the geometric and kinematic transformation which any continuous medium undergoes when traversed by a surge wave. It is accompanied by an appendix giving various matrix formulae relating to the deformation of continuous media, several of which are applied in this article.

Author's summary

4000:

Szaniawski, A. The structure of weak shock waves in real gases. Arch. Mech. Stos. **11** (1959), 173–192. (Polish and Russian summaries)

The purpose of the paper is to find some characteristic properties of the structure of weak shock waves in real gases. The functions representing the internal energy and the temperature are expanded in power series of the parameter $\varepsilon=M-1$ (M is the incoming Mach number). The medium satisfies approximately the equation of Clapeyron and its specific heat at constant volume is a variable. The relaxation phenomena are taken into account and each from n internal degrees of freedom possesses different relaxation time. Volume viscosity is also included. The fundamental dynamic equations governing the behavior of the medium are expanded into power series of the functions $(u-1)$, $(T-1)$, etc., assumed to represent small disturbances. After some simplifications, the power coefficients are equated to zero, giving the first approximations to $(u-1)$, specific entropy, etc. The thickness of the shock wave is used to define a proportionality coefficient used in the analysis. The procedure seems to be an interesting one and with a possibly suitable modification may be solved by means of computing devices.

M. Z. v. Krzywoblocki (Urbana, Ill.)

4001:

Szaniawski, Andrzej. The influences of molecular structure on the structure of a weak shock wave. Arch. Mech. Stos. 11 (1959), 317–332. (Polish and Russian summaries)

In his previous paper [reviewed above] the author found the first terms of the expansion of the function characterizing a shock wave in a power series of $\varepsilon = M-1$. The gas is viscous and heat conducting, is subject to relaxation phenomena and has temperature-variable specific heat. In that paper the author derived the equations for the thickness of the shock wave, velocity and specific entropy distributions. In the present paper the author considers the influence of the relaxation phenomena and the variation of the specific heat. In the equation of state and the expression for the specific heat there are added new terms which take into account three degrees of freedom of the translational motion, two degrees of freedom of the rotational motion and one degree of freedom of the vibration. With this, the author computes the expressions for the additional terms in the following cases: gas with no internal degrees of freedom, gas with internal degrees of freedom, diatomic gas and polyatomic gas. Disregarding the relaxation phenomena leads to errors greater than the quantities themselves; the relaxation time is greater than the time between collisions. The influence of the variation of the specific heat may be disregarded. The main difficulty in investigating the influence of the relaxation phenomena on the structure of a shock wave is the lack of knowledge of the precise values of the relaxation time and their dependence on the parameters of the thermodynamic state.

M. Z. v. Krzywoblocki (Urbana, Ill.)

4002:

Tartakovskii, B. D. On the diffraction of sound waves in converging beams. Soviet Physics. Acoust. 4 (4) (1958), 366–371 (355–360 Akust. Ž.).

"Approximate equations are found for the sound field near the focal point formed by a converging wave front which may be characterized by the nonuniform amplitude distribution over the front and by spherical aberration.

"The magnitude of nonuniformity in the amplitude over the wave front is estimated. The amplitudes can be significantly distinguished at a distance which exceeds the wavelength.

"Some special cases of the integral equations are investigated; in particular, the equations which describe the field on the axis of the wave beam at an arbitrary distance from the focus are considered and, in connection with these, the results obtained earlier by other authors are subjected to criticism." (Author's summary)

L. B. Felsen (Brooklyn, N.Y.)

4003:

Davies, D. R. On the calculation of eddy viscosity and heat transfer in a turbulent boundary layer near a rapidly rotating disk. Quart. J. Mech. Appl. Math. 12 (1959), 211–221.

The distribution of the radial component of shear stress in the above configuration is calculated from the equations of motion, assuming the velocity profiles to be of the 1/7th power law form suggested by von Karman [Z. Angew. Math. Mech. 1 (1921), 233–252; p. 245]. The distribution of eddy viscosity so found is plausible up to about $z/\delta = 0.02$ (δ = boundary layer thickness, z = height above disk) but decreases to zero at $z/\delta = 0.042$. The latter part of its behavior is rejected as physically unreasonable, and an alternative method of extrapolating the inner part of the distribution is proposed. Possible regions of laminar flow at the center of the disk and in the sublayer are excluded from consideration.

By supposing the thermal eddy diffusivity equal to the eddy viscosity so found, the heat transfer from the disk when the surface temperature is constant is calculated from the energy equation, by a method previously developed for flat plate boundary layers by the author and Bourne [Quart. J. Mech. Appl. Math. 9 (1956), 468–488; MR 18, 777]. Good agreement is found between the predicted Nusselt number and that found experimentally by Cobb and Saunders [Proc. Roy. Soc. London Ser. A 236 (1956), 343–351].

D. A. Spence (Ithaca, N.Y.)

4004:

Kulikovskii, A. G. Flow of conducting liquid past magnetized bodies. Dokl. Akad. Nauk SSSR (N.S.) 117 (1957), 199–202. (Russian)

The magnetic field extends from the magnetised body into a surrounding "cavern", occupied by fluid at rest, around which the remaining fluid flows. The fluid is perfectly conducting. Within the cavern, the magnetohydrostatic equations div $\mathbf{H}=0$, grad $p=(4\pi)^{-1}$ curl $\mathbf{H}\times\mathbf{H}$ are supplemented by curl curl $\mathbf{H}=0$; the latter condition is said to arise from considering perfect conductivity as a limiting case. Formulae are given for three special cases (i) flow round a cylindrical region containing a dipole distribution; (ii) supersonic flow past a wedge bearing a surface current, a certain minimum current being necessary for cavern formation; (iii) a corresponding problem for a cone.

F. V. Atkinson (Canberra)

4005:

Korobeinikov, V. P. Unidimensional automodel motions of a conducting gas in a magnetic field. Dokl. Akad. Nauk SSSR 121 (1958), 613–615. (Russian)

It is a question of non-stationary cylindrical or plane waves which evolve subject to similarity relationships. Starting with the four differential equations for v, ρ, p and $h=H^2/(8\pi)$, being functions of the position r and time t, these are transformed by the substitutions

$$v = rt^{-1}V,\quad \rho = ar^{-k-3}t^{-s}R,\quad p = ar^{-k-1}t^{-s-2}\mathscr{P},$$
$$h = ar^{-k-1}t^{-s-2}\mathscr{H},$$

where V, R, $\mathscr{P}$ and $\mathscr{H}$ depend only on a dimensionless variable $\lambda = r/(bt^{\delta})$. The resulting four ordinary differential equations can be reduced to two, using the integrals corresponding to the adiabatic and frozen-in properties; there may also be an energy integral, making possible a reduction to one differential equation. Shock waves are also considered. The article concludes with brief notes on special problems which might be considered in this way, such as the collision of cosmic masses, the motion of a piston in a gas, or an electrical discharge.

F. V. Atkinson (Canberra)

4006:

Kaempffer, F. A. Note on the self-energy of single roton states in quantum hydrodynamics. Canad. J. Phys. 32 (1954), 264–266.

Landau's theory of liquid He II (1941) with a sound foundation given by Ziman (1953) seems to justify the description of the behavior of He II in terms of phonons and rotons. The Hamiltonian in this case consists of three parts: H_{ph}, H_{rot} and H_{inter}. The author shows that the numerical results obtained by Ziman are not trustworthy, since H_{inter} contributes to the energy in second approximation a term of a high order of magnitude. Moreover, the values corrected by the author himself cannot be trusted either; any attempt at diagonalization of the Hamiltonian is unlikely to be successful if $H_{\text{ph}}+H_{\text{rot}}$ is taken as the unperturbed part of the Hamiltonian, since H_{inter} cannot be considered as small in the sense of perturbation theory.

M. Z. v. Krzywoblocki (Urbana, Ill.)

4007:

Horie, Chûji; and Ôsaka, Yukio. Note on Ziman's quantum hydrodynamics. Sci. Rep. Tôhoku Univ. Ser. I **38** (1954), 179–184.

Ziman [Proc. Roy. Soc. London Ser. A **219** (1953), 257–270] applied the quantization procedure to the rotational motion of a fluid and separated the Hamiltonian into terms corresponding to phonons, rotons and their interactions (following Landau's idea), but the interaction terms in his approach have only small contributions. Kaempffer [see review above] noticed that the interaction terms cannot be treated as small perturbations. Also Ito, quantizing the motion of the incompressible fluid, has shown that the energy spectrum does not include a finite energy gap as in Ziman theory. The authors apply the Bohm-Pines procedure to the Ziman Hamiltonian and after formal manipulations they obtain the roton spectrum expression in a form of a series whose initial terms differ from those of Ziman. The roton spectrum has a finite energy (as in Ziman's case) which is inversely proportional to the sound velocity, thus vanishing in the limiting case of an incompressible fluid-flow (Ito). Hence the appearance of a finite gap in the roton spectrum was confirmed. In the American school results of the same nature were obtained by Allcok and Kuper.

M. Z. v. Krzywoblocki (Urbana, Ill.)

4008:

Jankiewicz, Czesław. Das Variationsprinzip der relativistischen Hydrodynamik. Acta Phys. Polon. **18** (1959), 7–13.

A variational principle from which the equations of relativistic hydrodynamics are derived is described. The principle makes use of infinitesimal transformations of the coordinates. The equations of motion of a perfect fluid and of a viscous fluid are worked out as illustrations.

G. C. McVittie (Urbana, Ill.)

4009:

Matta, Gabriel. Application de la méthode de relaxation à l'étude des écoulements à surface libre en milieu poreux. C. R. Acad. Sci. Paris **244** (1957), 1720–1722.

4010:

Oroveanu, T.; et Pascal, H. Sur le mouvement d'un mélange compressible de liquide et de gaz à travers un milieu poreux. Rev. Méc. Appl. **2** (1957), no. 1, 93–99.

This study contains most of the results already obtained by the authors in two previous papers [Com. Acad. R. P. Romîne **5** (1955), 1311–1316; **6** (1956), 419–422].

A single equation is obtained for the flow of a mixture of a gas and a liquid based on a linear empirical law between ρ_l, the density of the liquid, and p, the pressure, and the assumption that

$$\mu = \mu_0 + a(p_0 - p)^s,$$

where μ is the mass ratio of equal volumes of the liquid and the gas and μ_0 corresponds to the pressure value, p_0. It is shown that there is a functional relation between p and ρ, the density of the medium.

Darcy's law is assumed and particular results are obtained for slow isothermal motions and for stationary states.

B. R. Seth (Kharagpur)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 4049.

4011:

Schnabel, B. Zur Beugungstheorie der Bildfehler. Optik **16** (1959), 449–456. (English summary)

The reasons behind the restrictions on the range of validity of present theories are discussed. On this basis a new theory is developed which has a significantly wider range of applications. This raises the question as to the value of the development by orthogonal functions from the point of view of exactness. One finds that the new theory makes possible the complete wave-optical calculation of an optical system. The case of vignetting can be treated without any additional assumptions in the new theory.

Author's summary

4012:

Abaunza, Arturo Arasti. Applications in optics of the ovals of Descartes. Rev. Acad. Ci. Zaragoza (2) **12** (1957), no. 2, 37–83. (Spanish)

The author treats the mathematical construction of cartesian surfaces (surfaces which make the rays from a single object point converge exactly to an image point).

He discusses their principal points, as well as their application in microscopy and as an auxiliary surface in order to invert images. He investigates the error against the sine condition and its effect on coma correction.

In the last chapter he discusses the methods of grinding or polishing cartesian surfaces.

M. Herzberger (Rochester, N.Y.)

4013:

Tserkovnikov, Iu. A. Stability of plasma in a strong magnetic field. Soviet Physics. JETP **5** (1957), 58–64.

The paper seems to be a study of the stability of an inhomogeneous plasma with respect to small perturbations in a strong magnetic field, carried out in a spirit similar to that of Brueckner and Watson [Phys. Rev. **102** (1956), 19–27]. In particular, the linearized Boltzmann equation with no collisions is used.

M. J. Moravcsik (Livermore, Calif.)

4014:

Moses, H. E. Solution of Maxwell's equations in terms of a spinor notation: the direct and inverse problem. Phys. Rev. (2) **113** (1959), 1670–1679.

The author solves Maxwell's equations in the form

given by him earlier [Nuovo Cimento (10) 7 (1958), suppl. 1–18; MR 20 #631], working with the spectral representation of the operator H_0 (Dirac Hamiltonian with mass term equal to zero). The orthonormal set used to obtain this representation is explicitly given as a set of four four-column vectors; they correspond to two circularly-polarized waves, a longitudinal field, and a column vector which vanishes if the divergence conditions in Maxwell's equations are satisfied. The method developed is applied to solve two classes of problems: (1) find the field if the sources are given; (2) find the sources if the field is given. As it is natural for physical reasons, the second problem has no unique solution, unless additional conditions are imposed. The author shows that a certain factorization property of the sources is such a sufficient condition. [The basic orthonormal set used is essentially the set of angular momentum eigenfunctions of the photon angular momentum operator. See, e.g., H. A. Kramers, *Die Grundlagen der Quantentheorie*, Vol. II, Akad. Verlagsges., Leipzig, 1938; pp. 424, 448; or A. I. Ahiezer and V. B. Beresteckiĭ, *Kvantovaya èlektrodinamika*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953; MR **16**, 431; see the English translation, *Quantum electrodynamics*, **AEC-TR-2876** (1953), Pt. I, p. 27.] *N. L. Balazs* (Princeton, N.J.)

4015:

Zlatev, Mintcho-P. Étude des champs électriques non potentiels. Rev. Gén. Élec. **68** (1959), 555–559.

L'auteur démontre, dans le présent article, une nouvelle relation entre le champ électrique tourbillonnaire induit **E** et le champ magnétique excitateur **H**. Cette démonstration est basée sur l'utilisation immédiate des deux lois fondamentales du champ électromagnétique sous forme intégrale: la loi d'Ampère et la loi généralisée de l'induction électromagnétique. Ainsi, connaissant l'un des vecteurs **E** ou **H**, on peut écrire directement l'expression de l'autre vecteur.

En vertu de cette relation, le champ **E** est déterminé immédiatement en fonction d'un vecteur paramétrique **X**. L'expression de **E** est analogue à celle de la loi de Biot-Savart-Laplace.

Les formules ainsi obtenues sont en principe valables pour une variation quelconque quasi stationnaire de l'induction magnétique en fonction du temps.

On démontre aussi que le champ électrique dans un circuit contenant un générateur est mixte tout le long de ce circuit, c'est-à-dire que la divergence et le rotationnel de ce champ sont égaux à zéro sur tout le circuit.

Résumé de l'auteur

4016:

Robieux, J. Lois générales de la liaison entre radiateurs d'ondes. Application aux ondes de surface et à la propagation. Ann. Radioélec. **14** (1959), 187–229. (English and German summaries)

This study concerns the determination of the general laws of linkage between wave radiators. The coefficient of transmission from a transmitter to a receiver has a general expression as a function of the field generated when a normalised wave carrying unit power is radiated by each aerial, taken in turn as a transmitting aerial. Another general expression determines the variation of the transmission coefficient with frequency.

These general expressions are rigorously demonstrated in two separate cases. In the first case a certain degree of homogeneity of the medium is assumed but little restriction is involved as regards the nature of the receiver. In the second case no restriction is involved as regards the nature of the medium, which may be homogeneous and anisotropic, but allows for the matching of the detector to the aerial by means of a transmission line, this condition always obtaining in a well-designed system.

This second case is of major interest because of its general application and the rigorous way in which it is obtained, starting from Maxwell equations and simple energy considerations. The Huyghens' principle and the Kottler formulae are merely special cases of this theorem. This general theorem can be most helpful in optics.

The relation between these theorems and the variational properties of the electromagnetic field are examined. The reasoning may be generalised. This generalization is demonstrated and applied in particular to wave mechanics and fluid mechanics.

These general theorems will be set out in Part I published in this issue. In a second and a third part, to be published in later issues, these theorems will be applied to the study of two special problems: the radiating properties of surface waves and phenomena of propagation by diffraction and scattering. *Author's summary*

4017:

Klante, Klaus. Zur Beugung skalarer Wellen am Rotations-Paraboloid. Ann. Physik (7) **3** (1959), 171–182.

This interesting paper concerns the diffraction by a paraboloid of revolution for a quantity satisfying the scalar Helmholtz equation $(\Delta+k^2)\phi=0$. A separation with the aid of parabolical coordinates ξ, η, φ leads to solutions

$$(\pm 2ik\xi)^{-1/2}M_{\sigma,p/2}(\pm 2ik\xi) \times (\mp 2ik\eta)^{-1/2}M_{\sigma,p/2}(\mp 2ik\eta)e^{\pm ip\varphi},$$

which determine a set of modes for integral values of p; M, which may also be replaced by the corresponding function W, represents a solution of Whittaker's differential equation for confluent hypergeometric functions. Each term of the Fourier series with respect to φ of the point-source solution $G=\exp(ikR)/(4\pi R)$ can be represented as a complex integral over the separation variable σ. This series reduces to a single integral over σ in cases of axial symmetry, e.g., for a point source on the axis of the paraboloid. The effect of such a point source is investigated in particular for the boundary condition of a vanishing normal derivative along the diffracting paraboloid $\eta=\eta_1$.

The solution corresponding to the latter situation can be represented by another complex integral over σ, which becomes rather simple along the diffraction surface itself. The integration path can be transformed into a contour and the integral thus proves to be equal to the sum of the residues at the enclosed poles of the integrand. The resulting residue series has the same character as the corresponding one in the diffraction theory for a spherical obstacle. Its individual terms can be interpreted as waves creeping along the diffracting surface. This becomes particularly clear from approximations based on asymptotic representations of the relevant Whittaker functions. For observation points in the shadow region the exponential

factor in these approximations indicates a direct propagation from the point source, along the tangent to the diffracting paraboloid, to the point of contact (horizon point), and a further propagation therefrom along the curved diffracting surface. The propagation along the latter section involves an exponential attenuation which can be described by an integral along the propagation path; for each mode its integrand can be expressed in terms of the local curvature of the diffracting surface, and of its first and second-order derivatives in the path direction. This suggests similar representations for other diffracting surfaces.

{Unfortunately the nomenclature is confusing in some parts. The symbol z is used for both the coordinate along the symmetry axis, and for the argument of the Whittaker functions. Also, G marks the free-space point source solution $\exp(ikR)/(4\pi R)$, as well as the complete solution for the boundary problem in question. The argument $-i\xi'$ of $m_\nu(p)$ in the first formula of section 4 has to be replaced by $-i\xi_0'$.} *H. Bremmer* (Eindhoven)

4018:

Elgot, Calvin C.; and Wright, Jesse B. Series-parallel graphs and lattices. Duke Math. J. **26** (1959), 325–338.

The set of series-parallel graphs is the smallest set which contains the graph having one edge and which is closed under the operations of placing two graphs in series and of placing two graphs in parallel. A graph G is a bridge if it satisfies both of the following conditions: (i) G has two distinguished vertices a and b such that every edge of G lies on a simple path from a to b; (ii) G has an edge π such that there exist simple paths from a to b which traverse π in each direction. Theorem: If a graph G satisfies (i), then G is series-parallel if and only if G is not a bridge. This result had long been assumed by electric circuit theorists, but this paper gives the first published proof of the theorem. The proof is stated in lattice-theoretic terminology. *E. F. Moore* (Murray Hill, N.J.)

QUANTUM MECHANICS

See also 3878, 3879, 4006, 4007, 4014.

4019:

Fischer, Otto F. Structurology and Kazuo Kondo's quantum-hydrodynamical analogy. Matrix Tensor Quart. **8** (1958), 92–101.

The first part of the paper contains a review of some work with K. Kondo in which analogies of turbulent flow with the Schroedinger and Maxwell equations have been attempted. In the second part the author connects these results to a formalism of his own devising. {Unfortunately this paper is too condensed to permit any statement by this reviewer as to the validity of this development.}

H. Feshbach (Cambridge, Mass.)

4020:

Fabre de la Ripelle, Michel. Sur la résolution des équations de la méthode des perturbations. Ann. Physique **3** (1958), 510–568.

The title of this paper gives a very poor description of its contents, which is an attempt at deriving from the Schrödinger equation of a quantum system a "master equation", i.e., a linear differential equation of first order for the occupation probabilities of a set of quantum states. This very important problem is dealt with only in an incomplete way, often difficult to follow, but the basic ideas are correct and a more thorough treatment would probably give a derivation of the master equation along lines similar to those adopted by the reviewer in work on the same problem [Physica **21** (1955) 517–540; MR **17**, 115]. The author of the present paper has obviously very little or no knowledge of the literature. Not only has the article no references at all, but it is clear that a study of the literature would have enabled the author to make a much better case for his own work, to stress its importance and, last but not least, to write a shorter and better paper free of many useless and often plainly wrong statements of detail. This is all the more surprising in that the paper, according to a footnote, is based on a doctoral dissertation defended in 1956 at the University of Paris.

L. Van Hove (Utrecht)

4021:

Tietz, T. Electron scattering cross sections with relativistic correction in the first Born approximation for Thomas-Fermi and Hartree potentials. Acta Phys. Acad. Sci. Hungar. **10** (1959), 19–27. (Russian summary)

Various approximations in closed form to the Thomas-Fermi function are used to derive analytical approximations to the (non-relativistic) scattering amplitude $f(\theta)$. Their quality is compared numerically in various tables which also contain numerical values relating to the Hartree field. An analytical approximation is given for the total cross-section valid for all velocities, and some numerical results based on it are given in the low velocity limit. *H. A. Buchdahl* (Princeton, N.J.)

4022:

Agranovič, Z. S.; and Marčenko, V. A. Construction of tensor forces from scattering data. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 1055–1058. (Russian)

The authors consider the problem of the determination of the potential V from scattering data for the problem

$$(*)\qquad Y''-[V(x)+6x^{-2}P]Y+\lambda^2Y=0,\quad 0<x<\infty,\quad Y(0)=0.$$

Here Y, V, P are all two-by-two matrices, with V being hermitian and satisfying, for some $\varepsilon>0$, $\int_0^\infty t^{1+\theta}|V(t)|dt<\infty$, $|\theta|<\varepsilon$, and $P=\begin{pmatrix}0&0\\0&1\end{pmatrix}$. The problem (*) has a continuous spectrum on the positive half-axis, and, possibly, a finite number of non-positive eigenvalues $0\geqq\lambda_1^2>\lambda_2^2>\cdots>\lambda_p^2$. For λ^2 in the spectrum there exist solutions $U(x,\lambda)$ of the differential equation such that $U(0,\lambda)=0$ and for which

$$\delta(x-y)\cdot I=\sum_{k=1}^{p}U(x,\lambda_k)U^*(y,\lambda_k)+\frac{1}{2\pi}\int_0^\infty U(x,\lambda)U^*(y,\lambda)d\lambda,$$

where δ is the δ-function, I=unit matrix. The matrices $U(x,\lambda)$ can be normalized so that the following asymptotic relations, as $x\to\infty$, are valid: $U(x,\lambda)\sim e^{i\lambda x}\cdot I-e^{i\lambda x}S(-\lambda)$ $(\lambda^2>0)$; $U(x,\lambda_k)\sim e^{-|\lambda_k|x}\cdot M_k$ $(\lambda_k^2<0)$; and if $\lambda_1=0$, then

$U(x, 0) \sim x^{-2} \cdot M_1$, $M_1 = mP$ $(m > 0)$. Here $S(\lambda)$ is a unitary matrix, called the scattering matrix; M_k is a hermitian matrix whose rank is equal to the multiplicity of λ_k^2. The dispersion data of (*) consists of S, the $\lambda_k^2 \leq 0$, and the M_k. The authors indicate (without proofs) a method of determining V from the scattering data. In some cases V is uniquely determined by the data, whereas in others only a one-parameter family of potentials is determined. Reference should be made to the authors' previous work on this subject [same Dokl. **113** (1957), 951–954; MR **19**, 746].

E. A. Coddington (Los Angeles, Calif.)

4023:

Goldberger, M. L.; and Treiman, S. B. A soluble problem in dispersion theory. Phys. Rev. (2) **113** (1959), 1663–1669.

These authors have previously [M. L. Goldberger and S. B. Treiman, same Rev. **110** (1958), 1178–1184; MR **19**, 1236] estimated the rate of decay of the charged pion, using coupled dispersion relations. Their result looked rather like an expression that could be obtained by a misapplication of perturbation theory. One of the purposes of the present paper, consequently, is to test the general principles of their approximation method by applying it to an exactly soluble model theory. For this purpose, they supplement the Lee model, with its strong transition $V \leftrightarrow N + \theta$, by the addition of a weak process $N + \theta \leftrightarrow N + \theta'$, thus allowing for the slow decay $V \to N + \theta'$. It is found that the dispersion relation method yields the correct result for this model; and the reason for the unexpected feature of the form of the answer is discussed.

J. C. Taylor (London)

4024:

Karplus, Robert; Sommerfield, Charles M.; and Wichmann, Eyvind H. Spectral representations in perturbation theory. II. Two-particle scattering. Phys. Rev. (2) **114** (1959), 376–382.

In a previous paper [same Rev. **111** (1958), 1187–1190; MR **20** #2996] the authors studied the triangle-diagram term in the three-point (vertex) function in a theory of interacting scalar fields. In this paper they discuss the square-diagram contribution to the four-point function which describes two-particle scattering. Of the six independent kinematic invariants, they fix the four masses of the incoming and outgoing particles and investigate the domain of analyticity of the scattering amplitude as a function of the remaining two real variables (linear combinations of (energy)^2 and (momentum transfer)^2 in any of the processes described). For fixed momentum transfer and certain values of the intermediate masses the energy threshold is "abnormal", i.e., less than the sum of the masses in the intermediate two-particle state. In such circumstances the usual derivations of dispersion relations cannot be used.

P. W. Higgs (London)

4025:

Sakurai, J. J. Symmetry laws and strong interactions. Phys. Rev. (2) **113** (1959), 1679–1692.

In this paper "an attempt is made to explore the possible connection between symmetry laws in internal space and symmetry laws in Lorentz space". In the framework of Yukawa interactions, CP invariant direct couplings or G invariant derivative couplings assure parity conservation in (charge-independent) pion-nucleon interactions. For K couplings parity conservation cannot be deduced from charge independence alone; a higher symmetry is then to be required of the K couplings. The "cosmic symmetry" is one such possibility; it is compared with the "global symmetry" proposed by Schwinger [Ann. Physics **2** (1957), 407–434; MR **19**, 1138] and by Gell-Mann [Phys. Rev. (2) **106** (1957), 1296–1300; MR **19**, 366]. Both these symmetries cannot exist at the same time since the "observed" symmetries are less than either of these higher symmetries; the author points out that the arguments in favour of the "global symmetry" model are very weak and hence he argues that the "cosmic symmetry" model is not ruled out. The development of these "cosmic" speculations relies heavily on a Lagrangian with distinct fields associated with all strongly interacting "elementary" particles.

E. C. G. Sudarshan (Rochester, N.Y.)

4026:

Sparnaay, M. J. On the additivity of London-Van der Waals forces. An extension of London's oscillator model. Physica **25** (1959), 217–231.

According to F. London's [Z. Phys. Chemie B **11** (1930), 222–251] harmonic-oscillator model, the energy of interaction between two neutral atoms, a large distance R apart, is given by $E_1 = -3h\nu\alpha^2/4R^6$, where h is Planck's constant, ν the frequency of oscillation and α the polarizability. The simplicity of this model invites generalization to systems of atoms. Therefore, consider two groups each of n identical atoms in which the distance between the two groups is large compared to the dimensions of either group. Additivity in the sense of the author means that the energy of interaction between the two groups is given by $E_n = n^2E_1$. Non-additivity is represented by $E_n = n^2E_1(1+f)$, where f depends on n and certain interaction parameters c_{ij} between atoms i and j of one group. Details of calculation are given for $n = 2$, 3, 5, and 7; the c_{ij} are restricted to dipole interaction and to nearest neighbours. The author's analysis involves transformation of quadratic forms and evaluation of determinants. It is shown that the non-additivity effect is of the order of from 10 to 30 per cent, depending on the geometrical configuration in the group relative to the whole system.

C. J. Bouwkamp (Eindhoven)

4027:

Petiau, Gérard. Sur un système d'équations d'ondes non linéaires généralisant les équations de la théorie du corpuscule de spin $\hbar/2$. C. R. Acad. Sci. Paris **248** (1959), 1620–1622.

4028:

van Hove, L. Many-body problems. Nederl. Tijdschr. Natuurk. **25** (1959), 227–239. (Dutch)

Expository lecture.

4029:

Valatin, J. G. Comments on the theory of superconductivity. Nuovo Cimento (10) **7** (1958), 843–857. (Italian summary)

The recent theory of superconductivity due to Bardeen, Cooper and Schrieffer [Phys. Rev. (2) **108** (1957), 1175–

1204; MR 20 #2196] is expressed in remarkably simple formal terms by the introduction of new creation and annihilation operators for the electrons. These operators are identical with those introduced independently by Bogoliubov in his discussion of the same problem [Soviet Physics JEPT 34 (7) (1958), 41–46; MR 20 #5670a]. Zero and positive temperatures are considered. As in the work of Bardeen and collaborators, and in contrast with Bogoliubov's approach, a variational method is adopted.
L. Van Hove (Utrecht)

4030:

Valatin, J. G.; and Butler, D. On the collective properties of a boson system. Nuovo Cimento (10) **10** (1958), 37–54. (Italian summary)

The discussion of a fermion system presented in the preceding paper is here transposed for a boson system, the formal method being the same but the physical contents being of course quite different. The results obtained reduce for weak coupling to those of the early work of Bogoliubov on the Bose gas [N. N. Bogoliubov, Acad. Sci. USSR. J. Phys. **11** (1947), 23–32; MR **9**, 168].
L. Van Hove (Utrecht)

RELATIVITY

See also 3949, 4008.

4031:

Scott, G. David. On solutions of the clock paradox. Amer. J. Phys. **27** (1959), 580–584.

The nature of the clock paradox is discussed and three solutions are referred to: (a) length contraction, (b) Doppler effects, (c) world lines in chronogeometry. The Special Theory of Relativity gives a complete explanation of the problem and it is pointed out how the use of the General Theory provides merely an additional solution with no physically new aspects. Variations in the clock problem are introduced to make clear the essential asymmetry which exists between the two clocks. *Author's summary*

4032:

Jankiewicz, Czesław. Über den Zusammenhang der Bewegungsgleichungen mit den Feldgleichungen. Acta Phys. Polon. **18** (1959), 21–35.

The author shows, by standard methods, that with distributed sources variation of one action integral leads to both the field equations and the equations of motion. As an example, he treats the combination of the gravitational with the electromagnetic field, plus "cold dust", i.e., matter which is characterized completely by a scalar mass density and a velocity field.
P. G. Bergmann (New York, N.Y.)

4033:

Higgs, P. W. Quadratic Lagrangians and general relativity. Nuovo Cimento (10) **11** (1959), 816–820. (Italian summary)

Field equations arising from the quadratic Lagrangians R^2, $R_{jk}R^{jk}$, $R_{jkst}R^{jkst}$ by independent variation of a symmetric tensor g_{jk} and a symmetric connection $\Gamma_{jk}{}^m$ have been given by Stephenson [Nuovo Cimento (10) **9** (1958), 263–269; MR **20** #7571]. In the present paper the equations arising from R^2 and from $R_{jk}R^{jk}$ are rewritten in the form $'R_{jk} = \lambda' g_{jk}$ (where $'R_{jk}$ belongs to $'g_{jk}$), together with algebraic relations between $'g_{jk}$ and g_{jk} involving an arbitrary function ψ. {It seems to the reviewer that the title of this paper is misleading. General relativity theory operates in a Riemann space, and when the Lagrangian is non-linear g_{jk} and $\Gamma_{jk}{}^m$ cannot be varied independently.}
H. A. Buchdahl (Princeton, N.J.)

4034:

Todorov, I. T. A uniqueness theorem for the wave equation. (On the discussion of V. A. Fok and F. I. Frankl'.) Uspehi Mat. Nauk (N.S.) **13** (1958), no. 2 (80), 211–213. (Russian)

In the polemics between V. A. Fok and F. I. Frankl' concerning the use of "harmonic coordinates" in the general theory of relativity, the latter showed by a counterexample that one of Fok's statements leads to a contradiction. The present paper indicates that this contradiction can be removed by a more careful formulation of the hypothesis. *G. Y. Rainich* (Notre Dame, Ind.)

4035a:

Fourès-Bruhat, Yvonne. Le problème de l'évolution dans le cas matière pure. C. R. Acad. Sci. Paris **246** (1958), 1809–1812.

4035b:

Fourès-Bruhat, Yvonne. Équations d'Helmholtz. Détermination des vitesses à partir des tourbillons. Problème d'évolution. C. R. Acad. Sci. Paris **246** (1958), 3319–3322.

These papers discuss cases (a) and (c) respectively, described in the following review.
N. L. Balazs (Princeton, N.J.)

4036:

Fourès-Bruhat, Yvonne. Théorèmes d'existence en mécanique des fluides relativistes. Bull. Soc. Math. France **86** (1958), 155–175.

This paper contains a collection of existence and uniqueness theorems in general relativity. First the author shows that it is sufficient to use isothermal coordinates in space-time; next she discusses four cases for which the resulting equation-systems are hyperbolic in the sense of Leray, and for which Cauchy's problem is well posed. Then the solutions exist, and are unique up to an arbitrary coordinate transformation, provided the initial data are correctly selected, are sufficiently differentiable, and are given on a hypersurface with a time-like normal. The following cases are discussed.

(a) Einstein field equations in the presence of incoherent matter ($T_{\alpha\beta} = \rho u_\alpha u_\beta$); equations of motion for matter; normalization condition on u_α. Fifteen unknowns: $g_{\alpha\beta}$ (10), metric tensor; u_α (4), four velocity; ρ (1), mass density. The well posed initial data are: $g_{\alpha\beta}$ with first and second derivatives, a time-like u_α, and ρ.

(b) Einstein field equations in the presence of charged incoherent matter and an electromagnetic field; equations of motion of matter; Maxwell's equations; normalization condition on u_α. Twenty unknowns: $g_{\alpha\beta}$ (10); u_α (4); ρ (1); ϕ_α (4), the four potential of the electromagnetic field; μ (1),

charge density. The well posed initial data are: $g_{\alpha\beta}$, ϕ_α with first derivatives; u_α, μ, ρ. (This case can also be obtained from the unified field theory of Jordan-Thiry.)

(c) Einstein field equations in the presence of an ideal fluid ($T_{\alpha\beta}=(\rho+p)u_\alpha u_\beta-pg_{\alpha\beta}$); equation of state $\rho=\rho(p)$; equation of motion for vorticity $\Omega_{\alpha\beta}=C_{\alpha,\beta}-C_{\beta,\alpha}$; equation of motion for pseudo-velocities $C_\alpha=fu_\alpha$ ($f=\exp\int dp/(p+\rho)$); normalization of C_α. Twenty-six unknowns: $g_{\alpha\beta}$ (10); $\Omega_{\alpha\beta}$ (10); C_α (4); ρ (1), density; p (1), pressure. The well posed initial data: $g_{\alpha\beta}$, with first derivatives; $\Omega_{\alpha\beta}$, C_α, with first and second derivatives; ρ, p.

(d) Einstein field in the presence of a charged ideal fluid and an electromagnetic field; equation of state; equations of motion for generalized vorticity $\pi_{\alpha\beta}=\Omega_{\alpha\beta}+(\mu f/\rho+p)F_{\alpha\beta}$; equation of motion for C_α; normalization on C_α; Maxwell's equations for the potentials ϕ_α. Thirty unknowns: $g_{\alpha\beta}$ (10); $\pi_{\alpha\beta}$ (10); C_α (4); ϕ_α (4); p (1); ρ (1). Well posed initial data: $g_{\alpha\beta}$, ϕ_α with first derivatives; $\pi_{\alpha\beta}$, C_α with first and second derivatives; ρ, p. *N. L. Balazs* (Princeton, N.J.)

4037:

Winterberg, F. Überprüfung der allgemeinen Relativitätstheorie durch Erdsatelliten. Nuovo Cimento (10) 8 (1958), 17–31.

The author suggests that the advent of artificial earth-satellites furnishes new possibilities for the verification of experimental results predicted by the general theory of relativity. On the advance of the perihelion due to relativistic effects, however, further effects resulting from the spheroidal shape of the earth are superimposed. A quantitative estimate of such effects is derived, which indicates that a test of this kind would be subject to serious difficulties. The red-shift is not to be measured directly: instead, it is suggested that atom clocks (e.g. Caesium clocks) based on the earth and on the satellite be used for this purpose. A quantitative estimate of the expected effect is found to be of an order of magnitude which could well be tested experimentally. It is found that neither the deflection of light by a gravitational field nor the dependence of the velocity of light on the gravitational potentials would give rise to effects which could be checked by experiment. Attention is paid to the additional gravitational field which is caused by the rotation of the earth according to the general theory of relativity: it is shown that this causes perturbations of the orbit which could in fact be tested. This is a point of special significance, as the quantitative treatment of this problem leads to extensions of the Schwarzschild solution and thus gives rise to possible tests of the general theory of relativity beyond the limitations of the Schwarzschild solution.

H. Rund (Durban)

4038:

Kalitzin, N. St. Über den Einfluss der Eigenrotation des Zentralkörpers auf die Bewegung der Satelliten nach der Einsteinschen Gravitationstheorie. Nuovo Cimento (10) 9 (1958), 365–374.

The influence of the rotation of a central astronomical body on the gravitational field and the corresponding effects on the motion of the satellites had been evaluated by Thirring and Lense by means of Einstein's approximate equations for the gravitational potential [Møller, *The theory of relativity*, Clarendon Press, Oxford, 1952; MR 14, 212; pp. 315, 317].

In the present paper this method is criticised on the grounds that (1) classical instead of relativistic perturbation methods were used, and that (2) the non-linearity of the effects involved was not taken into account. Thus the problem of evaluating the advance of the perihelion of a satellite of a rotating central body is treated from a fresh point of view based on a Lagrangian function introduced by V. A. Fok [*Teoriya prostranstva, vremeni i tyagoteniya*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955; MR 18, 445; p. 358 et seq.]. In addition to the usual expression for the advance of the perihelion a term due to the rotation of the central body is obtained, which differs from that found by Thirring and Lense solely by the form of its dependence on the angle i between the equatorial plane of the central body and the plane of the satellite orbit. According to the author the multiplicative factor $\{1-3\sin^2(i/2)\}$ in the formula of Thirring and Lense must be replaced by $\cos i$. These results are computed as far as terms in $(v/c)^2$. [See also F. Winterberg, reviewed above; p. 24].

H. Rund (Durban)

4039:

Kalitzin, N. St. Über die Bewegung der rotierenden Satelliten und Doppelsterne nach der Einsteinschen Gravitationstheorie. Nuovo Cimento (10) 11 (1959), 178–185. (Italian summary)

In a recent paper [reviewed above] the author had evaluated the advance of the perihelion of a satellite of negligible mass under the influence of the gravitational field of a rotating central body, his method being based on a Lagrangian introduced by V. A. Fok [*Teoriya prostranstva, vremeni i tyagoteniya*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955; MR 18, 445; p. 358 et seq.]. By means of an extension of this procedure the motion of a rotating satellite of finite mass about a rotating central body is considered in the present paper (again up to terms in $(v/c)^2$). This problem is of importance with regard to the relativistic theory of the motion of double stars. It is assumed that the mass-distributions of the bodies possess spherical symmetry, and that the radii of the bodies are small as compared with the distance between their centres. The author's expression for the advance of the periastron consists of a sum of three terms, of which the first is the counterpart of the expression found by Einstein for the advance of the perihelion of a planet, while the second corresponds to the expression due to Lense and Thirring (the latter of which is discussed in detail in the paper reviewed above). The third term furnishes a new additional effect on the advance of the periastron, being proportional to the sum of two terms, each of which consists of the product of the mass of one body and the specific angular momentum of the other. [Rev: Cf. also W. Tulczyjew, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 6 (1958), 645–651; MR 20 #7564].

H. Rund (Durban)

4040:

Popovici, Andrei. Théorie conforme relativiste-générale de corps tensoriels et spinoriels. I. Les équations du champ. Com. Acad. R. P. Romîne 8 (1958), 877–886. (Romanian. Russian and French summaries)

4041:

Costa de Beauregard, Olivier. L'effet gravitationnel de spin. C. R. Acad. Sci. Paris 246 (1958), 561–564.

4042:

Costa de Beauregard, Olivier. L'hypothèse de l'effet gravitationnel de spin. Cahiers de Phys. **12** (1958), 407–415.

The author postulates that a material medium endowed with spin gives rise to gravitational effects described by a non-symmetric tensor [cf. O. Costa de Beauregard, C. R. Acad. Sci. Paris **246** (1958), 237–240; MR **20** #6942; #4041 above; and D. W. Sciama, Proc. Cambridge Philos. Soc. **54** (1958), 72–80; MR **20** #727]. He outlines an experiment of a gyromagnetic type which would test this postulate. *D. W. Sciama* (London)

4043:

Petzold, Joachim. Die radialsymmetrische Lösung für skalare, masselose Mesonen in der projektiven Relativitätstheorie. Z. Naturf. **14a** (1959), 205–207.

The author solves the field equations of projective relativity for the metric tensor, the gravitational "constant" field and the scalar meson field of zero mass. The stress tensor occurring in the field equations is that due to the latter field. The fields are assumed to be spherically symmetric and stationary. It is shown that there does not exist a solution without a singularity. *A. H. Taub* (Urbana, Ill.)

4044:

Saint-Guilhem, René. Sur la forme mathématique des lois physiques. Changements d'unités, invariance des équations, analyse dimensionnelle. Rev. Gén. Élec. **68** (1959), 533–554.

L'auteur, en partant du principe selon lequel les unités de mesure peuvent être choisies arbitrairement, montre que l'on peut établir une théorie de l'invariance, dans une transformation affine, des relations mathématiques qui représentent les lois physiques.

Cette théorie générale qui utilise essentiellement la notion algébrique de groupe permet de lever les très nombreuses difficultés auxquelles donnent lieu les développements concernant les systèmes cohérents d'unités, l'homogénéité des relations physiques, l'analyse dimensionnelle, la similitude physique (théorie des modèles).

En particulier, un théorème fondamental nouveau, qui va plus loin que celui de Vaschy, conduit à une méthode d'analyse dimensionnelle parfaitement sûre qui donne d'emblée le résultat maximum susceptible d'être atteint par toute autre méthode. *Résumé de l'auteur*

ASTRONOMY

See also 3633, 3952.

4045:

★**Herrick, Samuel; Baker, Robert M. L., Jr.; and Hilton, Charles G. Gravitational and related constants for accurate space navigation.** Proceedings of the VIIIth International Astronautical Congress, Barcelona 1957, pp. 197–235. Springer-Verlag, Vienna, 1958. vii + 607 pp. $29.75.

4046:

Minin, I. N. On the theory of the diffusion of radiation in a semi-infinite medium. Soviet Physics. Dokl. **120 (3)** (1958), 535–537 (63–65 Dokl. Akad. Nauk SSSR).

The integral equation governing the luminance of a plane layer of infinitely large optical thickness with spherical scattering index with radiation sources depending only on the optical depth [V. V. Sobolov, Dokl. Akad. Nauk SSSR **116** (1957), 45–48] is solved by the Laplace transform method. For the case of pure scattering a table of values for related functions is given for various values of the optical depth. *R. G. Langebartel* (Urbana, Ill.)

4047:

Porfir'ev, V. V. On the law of rotation of a polytropic gas sphere. Astr. Ž. **36** (1959), 546–549. (Russian. English summary)

It is shown that a rotating polytropic gas sphere, in the absence of internal friction and circulating currents, can be in equilibrium only if the rotation is rigid. *Author's summary*

4048:

Simon, René. The hydromagnetic oscillations of an incompressible cylinder. Astrophys. J. **128** (1958), 375–383.

The small oscillations of an incompressible cylinder under its own gravity and in the presence of a uniform magnetic field parallel to its axis are considered. As distinguished from Chandrasekhar and Fermi [same J. **118** (1953), 116–141; MR **15**, 168], the author assumes that the magnetic field fills all space and not only the cylinder itself. This makes possible a fulfillment also of the boundary condition of the continuity of the magnetic pressure.

Although the equations of motion are not changed by the magnetic field, the boundary conditions give rise to a change in the constants, so that the frequency of the oscillations is augmented and the stability of the cylinder is increased. The presence of the external magnetic field makes this stabilization larger than that found by Chandrasekhar and Fermi.

Thereafter the cases of oscillations in only the radial and the azimuthal directions are treated, and it is found that an oscillation of this kind is independent of the magnetic field and always stable. *E. Lyttkens* (Uppsala)

4049:

Woltjer, L. The stability of force-free magnetic fields. Astrophys. J. **128** (1958), 384–391.

The force-free magnetic fields, the general aspects of which have been discussed by Chandrasekhar and Woltjer [Proc. Nat. Akad. Sci. U.S.A. **44** (1958), 285–289; MR **20** #5081] are characterized by the vanishing of the Lorenz force, so that curl $\mathbf{H}=\alpha\mathbf{H}$ and $\mathbf{H}\cdot\operatorname{grad}\alpha=0$.

Making use of a method, similar to that of Hain, Lüst and Schlüter [Z. Naturf. **12a** (1957), 833–841; MR **20** #608], the author shows that the condition for a stabilizing effect of the magnetic field with respect to a disturbance $\boldsymbol{\xi}$, vanishing at the boundary surface of the region considered, is

$$\int_V [\,|\operatorname{curl}(\boldsymbol{\xi}\times\mathbf{H})|^2-\alpha\boldsymbol{\xi}\times\mathbf{H}\cdot\operatorname{curl}(\boldsymbol{\xi}\times\mathbf{H})]d\tau\geqq 0,$$

the integral being extended over the volume V of the region considered.

The disturbance vector ξ is resolved into three components, one along **H** and the others along two vectors **a** and **b**. At first they are chosen so that $\xi \cdot \mathbf{b} = 0$, and from the formulae thus obtained the author shows that the stability condition is satisfied for radial disturbances and for disturbances having components only in the direction of the magnetic field and along a constant vector. In the general case, however, the stability condition derived for a suitably chosen rectangular coordinate system involves a term which can be negative.

In the axisymmetric case stability against axisymmetric disturbances is obtained when α is a constant. The generalization of Jeans' criterion is

$$\lambda = \frac{\pi}{G\rho_0}\,[c^2 + \tfrac{2}{3}\beta\langle V_A^2\rangle],$$

where c is the velocity of sound and β a constant depending on the orientation of the magnetic field and the wavelength of the disturbance considered. $\langle V_A^2 \rangle$ denotes the mean square of the Alfvén velocity, G the constant of gravity, ρ_0 the density and λ the wavelength of the disturbance considered. For the simplest mode of a sphere, Trehan [Astrophys. J. **126** (1957), 429–456, 601; MR **19**, 1125] has derived an expression from which the value $\tfrac{2}{3}\beta = 1.1$ can be obtained. *E. Lyttkens* (Uppsala)

4050:

Block, Lars. On the interplanetary gas and its magnetic field. Ark. Fys. **14** (1958), 179–193.

The models of interplanetary magnetic fields, given by Lüst and Schlüter [Z. Astrophys. **34** (1954), 263–282; **38** (1955), 190–211; MR **17**, 110] and Alfvén [Tellus **8** (1956), 1–12] are examined with respect to the corotation of the interplanetary matter at the Earth's distance from the sun at times of high solar activity and the low density of the interplanetary matter (about 0.1 particles/cm^3), for which the author quotes several reasons.

Outside a region of radius of about one-fifth of the Earth's distance from the sun, the corotation must be due to the magnetic field.

It seems necessary to make a compromise between the two fields mentioned above; viz., one which pulsates with the solar activity like that of Alfvén, but whose stationary state is closer to that of Lüst and Schlüter, a toroidal component being added. In this way it is possible to get a model in which the corotation of the interplanetary matter at the Earth's distance is present only when the solar activity is high, otherwise the corotation is not complete. When the solar activity is extremely low, the general galactic field dominates at the Earth's distance.
E. Lyttkens (Uppsala)

4051:

Neyman, Jerzy; and Scott, Elizabeth L. Statistical approach to problems of cosmology. J. Roy. Statist. Soc. Ser. B **20** (1958), 1–43.

A cosmology for an Euclidian space is built up as a stochastic process which is stationary in three dimensions, thus satisfying the cosmological principle. For the time-dependence, if such a dependence exists, a deterministic view is adopted.

After a review of the observational facts concerning galaxies, it is stated that the probabilistic approach is to be preferred because it can explain also the large variations. The mathematical model of the universe adopted is an extension of the earlier works of the same authors and C. D. Shane concerning the clustering of galaxies. The postulates and fundamental formulae are given in sections 4 and 5 of the paper; the dependence between the true spatial distribution and the two-dimensional distribution of the counted number of galaxies on a photographic plate being given in terms of probability generating functions.

After that the problem of discriminating between cosmologies postulating a universe stationary in time and those postulating a gradual thinning out of the matter is dealt with. In the latter case the apparent density of the distant nebulae is larger, which causes a diminishing of the correlation between the results of the counts for neighbouring fields. In this part the results given earlier by Neyman, Scott and Shane [Astrophys. J. **117** (1933) 92–112] are recalled.

The last section deals with selection bias in the observations concerning the luminosity-red shift relation, due to the fact that at large distances only intrinsically very bright nebulae can be observed and generally only very large clusters can be recognized as such.

In the discussion, following the paper, Professor M. S. Bartlett recommended a comparison with the multi-dimensional stochastic processes, studied by Whittle, Heine and others. Professor MacCrea questioned the assumption of independence between cluster centers and advocated an investigation of the ratio between the angular size of a cluster and the angular mean distance between neighbouring clusters for different values of the red shift. Mr. Walker called attention to the resemblance to the distribution of some plants in a field. Dr. W. L. Smith wanted a comparison with the results given by Agekeyan [Astr. Ž. **34** (1957), 371–378; MR **19**, 715] concerning the distribution of galaxies. In a later written contribution Mr. J. E. Moyal pointed out the applicability of the probability generating functional to the present problem, and derived some of the authors' formulae in that way.

The authors' reply included among other things an explanation of the bias in the observed angular distances between neighbouring nebulae for different values of the red shift due to the fact that intrinsically faint galaxies cannot be observed at large distances; an investigation concerning the rate of possible expansion of the universe; some comparison with Mr. Walker's results concerning the distribution of plants and a rather extensive discussion of Agekeyan's results. *E. Lyttkens* (Uppsala)

GEOPHYSICS

4052:

Jordán Díaz, Plácido. On the Jacobian of two functions: its interpretation in meteorology. Rev. Soc. Cubana Ci. Fís. Mat. **4** (1957/58), 87–90. (Spanish)

Les équations différentielles qui se présentent dans la théorie de la prévision du temps par les méthodes numériques introduisent la notion d'un Jacobien de deux fonctions, qui représentent certaines propriétés de l'atmosphère (p.e., température, pression, humidité, etc.). On trouve

rarement dans les traités météorologiques une interprétation de ce Jacobien. Dans sa courte note l'auteur précise cette notion. *M. Kiveliovitch* (Paris)

4053:

Piddington, J. H. The transmission of geomagnetic disturbances through the atmosphere and interplanetary space. Geophys. J. **2** (1959), 173–189.

The theory of the propagation of slowly varying electromagnetic disturbances through partially ionized gas is developed and applied to the Earth's atmosphere and interplanetary space. The medium must be regarded as two separate, co-existing gases, and electron-ion plasma and neutral atoms which move to some extent independently. Quantitative results are given for a model atmosphere out to several Earth radii.

From the author's summary

OPERATIONS RESEARCH, ECONOMETRICS, GAMES

4054:

Konijn, H. S. A restatement of the conditions for identifiability in complete systems of linear difference equations. Metroecon. **10** (1958), 182–190.

The author restates and proves the conditions given by Koopmans, Rubin, and Leipnik [*Statistical inference in dynamic economic models*, pp. 53–237, Wiley, New York, 1950; MR **12**, 431] and restated by W. C. Hood and T. C. Koopmans [*Studies in Econometric Method*, Wiley, New York, 1953; MR **15**, 812]. *H. Rubin* (East Lansing, Mich.)

4055:

★**Arrow, Kenneth J.; and Hoffenberg, Marvin. A time series analysis of interindustry demands.** Contributions to Economic Analysis, XVII. North-Holland Publishing Co., Amsterdam, 1959. ix + 292 pp. $7.25.

This six-year research project explores data from two sources: (A) Input-output table for 43 production sectors and 22 final demands, obtained by aggregation of the 450×450 sector table compiled for 1947 by the U.S. Dept. of Labor. (B) Yearly time series 1929–1950 for the 43 production volumes $x_i(t)$ and the 22 final demands $f_i(t)$.—The analysis is in two steps: (i) For four production sectors, time series for the yearly output balance are calculated as residuals $r_i(t)=x_i(t)-\sum_j \bar{a}_{ij}x_j(t)-f_i(t)$, assuming that (*) the input-output coefficients $\bar{a}_{ij}$ for 1947 are temporally stable. Assumption (*) is rejected on the basis of variance ratio tests in multiple regression analyses of $r_i(t)$ in terms of three alternative sets of 12 out of 19 explanatory factors. (ii) The trends in the input-output coefficients are explained by a linear model $a_{ij}(t)=\bar{a}_{ij}+\sum_k \lambda_{ijk}\varphi_k(t)$ involving four specified trend factors $\varphi_k(t)$. The coefficients λ_{ijk} are estimated by (a) simultaneous equations methods, which proved unusable, and (b) a linear programming method, minimizing $\sum_t |x_i(t)-\sum_j a_{ij}(t)x_j(t)-f_i(t)|$.—The emphasis on data reliability will be noted. In their cautious conclusions the authors stress the fargoing multicollinearity of the time series data, a feature of regularity which the input-output approach does not exploit in the model construction. *H. Wold* (Uppsala)

4056:

Basmann, R. L. The computation of generalized classical estimates of coefficients in a structural equation. Econometrica **27** (1959), 72–81.

In a classic article, Girshick and Haavelmo [Econometrica, **15** (1947), 79–110] illustrated the computation of the "limited information single equation" method (L.I.S.E.) for estimating the coefficients in simultaneous structural equations. Their illustration involved a five-equation system. In the present paper, the computation of the "generalized classical estimate" (G.C.) is illustrated for the same system and data. The L.I.S.E. and G.C. results are compared. {There is an obvious typographical error in the labeling of the top of the M_{yy} matrix on page 75.} *H. Chernoff* (Stanford, Calif.)

4057:

Friedman, Lawrence. Competitive coefficients. Management Sci. **5** (1959), 404–409.

The value relationships among the competitors in a competitive situation are described by means of competitive coefficients. The competitive coefficient defines the indirect gain or loss to a competitor as a result of losses or gains by the other competitors. An example is given to illustrate the use of the coefficients in a game situation. *Author's summary*

4058:

★**Batchelor, James H. Operations research: An annotated bibliography.** 2nd ed. Saint Louis University Press, St. Louis, Mo., 1959. x + 866 pp.

This is a new edition of an earlier bibliography (1952), brought up to the end of 1957. Considerable new material has been added. Abstracts of papers and books and reports on operations research are listed by author or editor. The coverage is quite complete, except in some specialized fields, such as linear programming, which are already covered by other bibliographies. There is also an index, according to title, but no analytic subject index.

P. M. Morse (Cambridge, Mass.)

4059:

Chuard, Philippe. Remarques sur le calcul des primes pour rentes de survie. Mitt. Verein. Schweiz. Versich.-Math. **59** (1959), 99–117. (German, Italian and English summaries)

Three simplified methods in calculating the premiums "have been used for the Swiss group insurance tariff in force since 1953 as well as for the individual annuity tariffs which have been based on the former one. In the present paper the author has analysed the errors due to the application of these approximate methods and has endeavoured to determine the limits of their applicability." (From the author's summary) *K. Medin* (Uppsala)

4060:

Levy, Joel. Further notes on the loss resulting from the use of incorrect data in computing an optimal inventory policy. Naval Res. Logist. Quart. **6** (1959), 25–31.

The author restates the optimal inventory policy and its cost for a model with order costs and penalty for depletion costs directly proportional to order quantity and shortage quantity respectively. When this policy is applied to

erroneous data it is no longer optimal. An expression for the resulting loss is derived for two cases: (1) The penalty per unit shortage is actually p but incorrectly assumed to be $\bar{p}$. (2) The appropriate discount rate is a but incorrectly assumed to be $\bar{a}$. *W. W. Leutert* (New York, N.Y.)

4061:

Volkmann, Bodo. Gewinnmengen. Arch. Math. **10** (1959), 235–240.

A solution is given for the nested interval game of Mazur, previously solved by Banach [cf. Mycielski, Świerczkowski, and Zięba, Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 485–488; MR **19**, 232; see also a paper by the reviewer, *Contributions to the theory of games*, vol. 3, pp. 159–163, Princeton Univ. Press, 1957; MR **20** #264]. The author's solution is in slightly different terms. Let U be a family of closed intervals densely distributed in $[0, 1]$, and let $L(U)$ denote the set of all points that can be represented as the intersection of a decreasing sequence of intervals belonging to U. Then a set $M \subset [0, 1]$ is a winning set for the second player if and only if it contains $L(U)$ for some U. {The author's proof of the necessity of this condition is incorrect. In fact, $L(U) = [0, 1]$ for the family $U = \mathfrak{R}^*$ which he constructs on p. 237. Nevertheless, the theorem is true.} Examples of winning sets of arbitrary Hausdorff dimension α, $0 \leqq \alpha \leqq 1$, are given. E.g., let $\nu(x, i)$ be the frequency with which 0 occurs among the first i digits in the binary development of x, and let $V(x)$ be the set of limit points of $\nu(x, i)$. Then the set $\mathrm{M} = \{x : V(x) = [0, 1]\}$ is a winning set of Hausdorff dimension 0.
J. C. Oxtoby (Bryn Mawr, Pa.)

4062:

★Harsanyi, John C. A bargaining model for the cooperative n-person game. Contributions to the theory of games, Vol. IV, pp. 325–355. Annals of Mathematics Studies, no. 40. Princeton University Press, Princeton, N.J., 1959. xi+453 pp. $6.00.

The author gives a determinate solution to an n-person cooperative game in normal-form which defines "the actual payoffs that the players of the game, if they act in accord with certain rationality postulates, tend to agree upon in bargaining." The first set of five rationality postulates can be grouped as follows: Given that a set of k players by joint unanimous action can achieve any payoff vector in a compact and convex prospect space, but lacking this unanimity, can only receive a unique "disagreement-payoff" vector, then they will jointly agree on the Nash bargaining point (directly generalized from 2 to an arbitrary $k \geqq 2$). The outcome of the game is then postulated to be determined by a network of interacting agreements made by the $2^n - 1$ non-empty subsets of the n-players. Each subset, called a syndicate, first announces its own threat strategy and then, as a function of all the threat strategies, announces a compatible dividend structure for its members. A particular set of threat and dividend strategies for the syndicates which is in equilibrium is singled out and the payoff to an individual is the sum of all his dividend shares taken over all supersets of his one-element set. The theory does not involve interpersonal comparisons of utilities but in an appendix this case is considered and analogies to Shapley value are indicated. Further optimal properties are still to be investigated. *H. Raiffa* (Cambridge, Mass.)

BIOLOGY AND SOCIOLOGY

See 3929.

INFORMATION AND COMMUNICATION THEORY

See 3314, 3691.

SERVOMECHANISMS AND CONTROL

See also 3331, 3437.

4063:

Carter, D. S. A minimum-maximum problem for differential expressions. Canad. J. Math. **9** (1957), 132–140.

Let the n by n matrices $A(t)$, $A^{-1}(t)$, $B(t)$ and the n-vector $c(t)$ be Lebesgue summable on $[a, b]$. Let η_a, η_b be constant n-vectors and $J^1, \cdots, J^n$ measurable subsets of $[a, b]$. Define the norm $\|f\|$ of an essentially bounded vector function $f(t) = (f^1, \cdots, f^n)$ to be $\max(\text{ess sup}\,|f^i(t)|)$. Let X be the set of absolutely continuous vector functions $x = x(t)$ on $[a, b]$ such that $x(a) = \eta_a$, $x(b) = \eta_b$, that $Lx = A[x' + Bx + c]$ is essentially bounded and that the ith component of Lx vanishes almost everywhere on $I - J^i$. The author considers the question of the existence of an $x_0 \in X$ satisfying $\|Lx_0\| = \inf\{\|Lx\| : x \in X\}$. He shows that if the problem is consistent, then there exists at least one solution x_0 and, for any solution x_0, the absolute value of the ith component of Lx_0 is constant on a certain specified subset of J^i. *P. Hartman* (Baltimore, Md.)

4064:

Carter, D. S. Correction to a minimum-maximum problem for differential expressions. Canad. J. Math. **9** (1957), 226.

The author corrects three misprints in his paper reviewed above and states a relationship between his results and a paper on a certain control problem [R. Bellman, I. Glicksburg and O. Gross, Quart. Appl. Math. **14** (1956), 11–18; MR **17**, 1206]. In this connection, see the recent paper by J. P. LaSalle reviewed below.
P. Hartman (Baltimore, Md.)

4065:

LaSalle, J. P. Time optimal control systems. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 573–577.

Let $A(t)$, $B(t)$ be continuous $n \times n$, $n \times r$ matrices, respectively, $f(t)$ a continuous n vector for $t \geqq 0$, x an n vector and u an r vector. Let Ω be the set of steering functions $u(t) = (u_1(t), \cdots, u_r(t))$, where $u_i(t)$ is a measurable function for $t \geqq 0$ satisfying $|u_i(t)| \leqq 1$; Ω_0 the subset of Ω consisting of bang-bang steering functions, those $u(t)$ for which $|u_i(t)| \equiv 1$. Let x_0 and a path $x = z(t)$ be given, let $x = x(t; u)$ be the solution of (1) $x' = A(t)x + B(t)u(t) + f(t)$ satisfying $x = x_0$ at $t = 0$; a $u^* \in \Omega$ is called optimal relative to Ω if, for some $t^* > 0$, $x(t^*; u^*) = z(t^*)$ and $x(t; u) \neq z(t)$ for $0 < t < t^*$ and all $u \in \Omega$. The author announces several results, among which are the following. (I) If $u^* \in \Omega_0$ is optimal relative to Ω_0, then it is optimal relative to Ω.

(II) If Ω contains an optimal steering function, then Ω_0 contains one. (III) If, for some $u \in \Omega$, $x(t;u)=z(t)$ for some $t>0$, then Ω contains an optimal steering function u^* which is necessarily of the form (2) $u^*(t)=\mathrm{sgn}\,(yX^{-1}(t)B(t))$, where $X(t)$ is a fundamental matrix solution of $x'=A(t)x$, y is an n-dimensional (row) vector and $\mathrm{sgn}\,(v_1, \cdots, v_r)=(\mathrm{sgn}\,v_1, \cdots, \mathrm{sgn}\,v_r)$. A condition on $X^{-1}(t)B(t)$, called normality, implies that (2) is unique. Other theorems deal with the case $f(t)\equiv 0$ and $z(t)\equiv 0$; e.g., a partial converse of (III) is obtained in this case.

P. Hartman (Baltimore, Md.)

4066:

Gamkrelidze, R. V. Optimum-rate processes with bounded phase coordinates. Dokl. Akad. Nauk SSSR **125** (1959), 475–478. (Russian)

The problem of transforming a system from one state into another in minimum time, or at minimum cost in resources, has attracted a considerable amount of attention in recent years [see the article by J. P. LaSalle, reviewed above, for the most general results concerning linear systems, extending earlier results of Bellman, Glicksberg and Gross, Quart. Appl. Math. **14** (1956), 11–18; MR **17**, 1206; and for a history and analysis of other results].

In this paper, the author considers nonlinear systems and sets up a Hamilton-Jacobi formalism.

R. Bellman (Santa Monica, Calif.)

4067:

Sawaragi, Yoshikazu; and Sunahara, Yoshifumi. On the statistical studies of response of automatic control systems with a non-linear element of zero-memory type. II. Statistical evaluation of response of servosystems with a non-linear element of zero-memory type to a sinusoidal signal and a Gaussian disturbance. Tech. Rep. Engrg. Res. Inst. Kyoto Univ. **8** (1958), 195–218.

The preceding article in this series [see same Rep. **8** (1958), 95–126; MR **20** #793] deals with the approximate representation of a nonlinear dynamical element, of the zero-memory type, by a pseudo-linear element characterized by certain equivalent gains depending on the applied signals. In this second article such a nonlinear element is considered as an element in an otherwise linear control system, and it is assumed that the system is subjected simultaneously to a sinusoidal input signal and to a Gaussian noise. The objective is the determination of statistical properties (e.g., mean values and power spectra) of the resulting signals at various points of the system. The nonlinear element being represented by its pseudo-linear approximation, and the equivalent gains being defined statistically as in the preceding article, this determination can be carried through in a straightforward way. The essentials of the procedure are already familiar from prior papers dealing with simpler situations [see, for instance, an article by R. C. Boonton, Jr., *Proceedings of the symposium on nonlinear circuit analysis, New York, 1953*, pp. 369–391, Polytechnic Inst. of Brooklyn, New York, 1953; MR **16**, 1036]; and the present discussion is noteworthy chiefly for the somewhat complicated case considered, and for the elaborate detail with which the general theory is treated. Aside from one illustrative example, the present article is devoted exclusively to the general formulae and methods. Applications of the theory to some particular problems are discussed in the following article.

L. A. MacColl (New York, N.Y.)

4068:

Sawaragi, Yoshikazu; and Sunahara, Yoshifumi. The statistical studies on the response of automatic control systems with a non-linear element of zero-memory type. III. Tech. Rep. Engrg. Res. Inst. Kyoto Univ. **9** (1959), 77–96.

This is a sequel to the paper reviewed above, and is concerned with a few applications of the general theory. These applications, all of which relate to questions of stability, are not explained very clearly. The authors describe certain experimental studies, the results of which are interpreted as confirming the predictions of the theory.

L. A. MacColl (New York, N.Y.)

4069:

Krasovskiĭ, A. A. On synthesis of impulsive compensation devices of servosystems. Avtomat. i Telemeh. **20** (1959), 729–739. (Russian. English summary)

Optimum distributions of closed loop servosystems weighting coefficients are determined in the cases both of a slowly changing useful signal with arbitrary stationary random noise and of a stationary random useful signal. Formulae are deduced for linear discrete compensation device coefficients corresponding to given closed-loop servosystems weighting coefficients. *Author's summary*

4070:

Karpova, N. A. Contact schemes for monotonic functions. Dokl. Akad. Nauk SSSR **123** (1958), 25–27. (Russian)

Let $\alpha=(\alpha_1, \cdots, \alpha_n)$ and $\beta=(\beta_1, \cdots, \beta_n)$ be ordered n-tuples of zeros and ones. The relation $\alpha\prec\beta$ or, more explicitly, $(\alpha_1, \cdots, \alpha_n)\prec(\beta_1, \cdots, \beta_n)$ means that $\alpha_i \leq \beta_i$, $1\leq i\leq n$. A Boolean function $f(x_1, \cdots, x_n)$ is monotone if the inequality $f(\alpha_1, \cdots, \alpha_n)\leq f(\beta_1, \cdots, \beta_n)$ is satisfied for every α and β satisfying $\alpha\prec\beta$. Let $L^+(f)$ be the number of contacts in a minimal realization of a monotone function f in the form of a network comprising only make contacts. Let $L(f)$ be the number of contacts in a minimal realization of f in the form of a network comprising both make and break contacts. The main result of the paper is a constructive demonstration of the existence of a sequence of monotone functions $\{f_n\}$ (f_n denoting an n-place function) such that $L^+(f_n)/L(f_n)\geq \frac{9}{8}$ for large n.

L. A. Zadeh (Berkeley, Calif.)

HISTORY AND BIOGRAPHY

4071:

Huber, Peter. Bemerkungen über mathematische Keilschrifttexte. Enseignement Math. (2) **3** (1957), 19–27.

4072:

Bruins, E. M. Pythagorean triads in Babylonian mathematics. Math. Gaz. **41** (1957), 25–28.

4073:

Neugebauer, O. "Saros" and lunar velocity in Babylonian astronomy. Mat.-Fys. Medd. Danske Vid. Selsk. **31** (1957), no. 4, 21 pp. (2 plates)

4074:
Stoltenberg, Hans L. Minoische Bruchzahlzeichen und ihre Selbständigkeit. Nachr. Giessen. Hochschulges. **25** (1956), 130–137.

4075:
Biermann, Kurt-R.; und Mau, Jürgen. Überprüfung einer frühen Anwendung der Kombinatorik in der Logik. J. Symb. Logic **23** (1958), 129–132.

Plutarch quotes Hipparch as having computed the numbers of positive and negative conjunctions of ten propositions. In the interpretation proposed by the authors both numbers appear to be false. Probably Hipparch's numbers should be interpreted in another way. In any case it would be advisable to check some manuscripts of Plutarch. *H. Freudenthal* (*Utrecht*)

4076:
Busulini, Bruno. Introduzione a una storia e filosofia del calcolo infinitesimale. Ann. Univ. Ferrara. Sez. VII. (N.S.) **7** (1957/58), 9–52. (French summary)

4077:
Boyer, C. B. Descartes and the geometrization of algebra. Amer. Math. Monthly **66** (1959), 390–393.

The author wishes to correct a common misstatement to the effect that Descartes arithmetized or algebraicized geometry. The author seeks to show that the purpose of Descartes' *La géométrie* could with equal validity be described as the translation of algebraic operations into the language of geometry or the geometrization of algebra. He interprets the purpose of Books I and III of *La géométrie* to be just this, though Book II contains, as he points out, the foundation of modern analytic geometry.

The history of the relationship between algebra and geometry from Greek times on is very briefly recalled in order to make clear the status of the two subjects in Descartes' times.

Toward the end of the article the author states that the purpose of *La géométrie* is brought out in the third book and "is the geometric construction of the roots of polynomial equations. . . . While Descartes did indeed convert geometric problems into the language of algebra this was not an end in itself; it was but an intermediate step in determining the most appropriate geometrical construction." This interpretation is questionable. Descartes did propose to classify curves in accordance with degree in the expectation that algebraic methods suited to each degree would be found. Hence the facilitation of the study of geometry was certainly an objective, as the author grants in other statements. Secondly it is doubtful that one can find a unity of purpose in all three books of *La géométrie*. Finally the statements found often in histories of mathematics that Descartes arithmetized geometry are meant to describe the ultimate effect of Descartes' work rather than his intentions. *M. Kline* (Aachen)

4078:
Murata, Tamotsu. French empiricism. II, III. Comment. Math. Univ. St. Paul. **7** (1959), 57–63, 99–116.

[For part I, see same Comment. **6** (1958), 93–114; MR **19**, 1247.] Here the author defends the opinion that Borel uses the term "effectif" in the sense of "efficacious for the construction of the important parts of mathematics", not in that of "concretely constructible". *A. Heyting* (Amsterdam)

4079:
Marković, Željko. Obituary: Radovan Vernić. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II **13** (1958), 287–290. (Serbo-Croatian)

Includes scientific bibliography.

4080:
Popovici, Andrei. Albert Einstein. Gaz. Mat. Fiz. Ser. A **9** (**62**) (1957), 207–213. (Romanian)

4081:
Iacob, Caius. Obituary: Nicolae Ciorănescu. Gaz. Mat. Fiz. Ser. A **9** (**62**) (1957), 214. (Romanian)

4082:
Nevanlinna, Rolf; und Wittich, Hans. Egon Ullrich in memoriam. Jber. Deutsch. Math. Verein. **61** (1958), Abt. 1, 57–65.

Personal biography by the first author, scientific by the second.

4083:
König, Heinz. Obituary: Hilbert Bilharz. Jber. Deutsch. Math. Verein. **61** (1958), Abt. 1, 97–103.

Includes bibliography.

MISCELLANEOUS

4084:
★Otto, Edward. Matematyka: Podręcznik dla wydziałów budowlanych politechniki. [Mathematics: A text-book for civil engineering students.] Państwowe Wydawnictwo Naukowe, Warsaw, 1959. 500 pp. zł. 36.00.

This is a carefully written text for engineers, covering plane and solid analytic geometry, calculus and introductory chapters on ordinary and partial differential equations and functions of a complex variable. Included are: proof of the Cauchy-Peano existence theorem, a discussion of potentials, proof of the fundamental theorem of algebra, brief classification of partial differential equations, etc. Many applications to physics, chemistry and technology are discussed, there are problems at the end of most sections and a comprehensive index is included. *Z. A. Melzak* (Montreal, P.Q.)

4085:
★Carroll, Lewis [Dodgson, C. L.]. Mathematical recreations of Lewis Carroll. Vol. 1: Symbolic logic and The game of logic. (Both books bound as one.) Dover Publications, Inc., and Berkeley Enterprises, Inc., New York, 1958. xxxi + 199 + ix + 96 pp. $1.50.

An unaltered reproduction of *Symbolic logic I: Elementary* [4th ed. Macmillan, London-New York, 1897] and *The game of logic* [Macmillan, London-New York, 1886].

4086:

★**Carroll, Lewis [Dodgson, C. L.]. Mathematical recreations of Lewis Carroll. Vol. 2: Pillow problems and A tangled tale.** (Both books bound as one.) Dover Publications, Inc., New York, 1958. xxii + 109 + vii + 152 pp. $1.50.

An unaltered reproduction of the 4th edition of *Pillow problems thought out during wakeful hours* [Macmillan, London, 1893] and *A tangled tale* [Macmillan, London, 1885].

4087:

★**Dudeney, Henry Ernest. Amusements in mathematics.** Dover Publications, Inc., New York, 1959. xii + 258 pp. $1.25.

An unaltered republication of the first edition [Nelson, London-New York, 1917], except for the addition of notes on British coins and stamps, and on cricket.

4088:

★**Dudeney, Henry Ernest. The Canterbury puzzles and other curious problems.** 4th ed. Dover Publications, Inc., New York, 1959. 255 pp. $1.25.

An unaltered republication of the fourth edition [2nd ed., Nelson, London, 1919] except for the addition of a note on British coins and stamps.

4057:
Dudeney, Henry Ernest. Amusements in mathematics. Dover Publications, Inc., New York, 1958. xii + 258 pp. $1.35.
An unaltered republication of the first edition [Nelson, London-New York, 1917], except for the addition of notes on British coins and stamps, and on cricket.

4058:
Dudeney, Henry Ernest. The Canterbury puzzles and other curious problems. 4th ed. Dover Publications, Inc., New York, 1958. xvi + 255 pp. $1.25.
An unabridged republication of the fourth edition (2nd ed. Nelson, London, 1919) except for the addition of a note on British coins and stamps.

An unaltered reproduction of Symbolic logic I: Elementary (4th ed. Macmillan, London-New York, 1897) and The game of logic (Macmillan, London-New York, 1886).

[illegible]:
Carroll, Lewis (Dodgson, C. L.). Mathematical recreations of Lewis Carroll. Vol. 2: Pillow problems and A tangled tale. (Both books bound as one.) Dover Publications, Inc., New York, 1958. xxii + 109 + 152 pp. $1.50.
An unaltered reproduction of the 4th edition of Pillow problems thought out during wakeful hours (Macmillan, London, 1895) and A tangled tale (Macmillan, London, 1886).

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